INTEGRAL MODEL OF EDDY CURRENTS IN NONMAGNETIC STRUCTURES

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SUMMARY

Even when most of eddy current problems can succesfully be solved by differential techniques (particularly by the finite element methods), there exists a group of tasks where their application may cause various complications. Mentioned can be presence of geometrically incommensurable subdomains in the investigated area, ignorance of the boundary conditions, movement of some parts in the system etc. In such cases integral models may sometimes prove to be more advantageous. The paper presents the basic integral model of eddy current given by a system of the second-kind Fredholm integral equation and possibilities of using integral schemes of higher order of accuracy. The theoretical considerations are supplemented by two illustrative examples in 2D whose results are discussed and compared with values obtained by other ways.

Keywords: *magnetic fields, eddy currents, integral equations, numerical methods*

1. INTRODUCTION

Investigation of eddy currents in nonmagnetic structures represents a problem that is nowadays mostly solved by various differential techniques (mostly FEM). Despite this fact, however, there exists a group of tasks whose processing by these techniques is rather complicated and from time to time can fail. Mentioned can be arrangements with elements that are geometrically incommensurable (thin conductors versus air) or that move with respect to one another, which requires remeshing of the whole domain at each time step. Difficulties may also be caused by ignorance of boundary conditions (which is quite typical for these tasks), but they can often be overcome by using artificial boundary (with a physically real boundary condition) that is sufficiently distant from the area of our interest.

An alternative for solution of such problems (as far as they are linear) is integral approach. Its principal ideas are not new, they were dealt with by a number of authors (see, for example [1], [2] and [3]), but up to now it was only rarely used for tackling eddy current problems because of several drawbacks. The principal disadvantage consists in the fact that the corresponding numerical schemes lead to systems with fully or strongly populated matrices and further difficulties may appear in association with numerical evaluation of various proper and improper integrals (whose values have to be determined with a sufficiently high accuracy).

The basic mathematical model of the integral approach is given by a system of the second-kind Fredholm integral equations. And this system can be processed in several ways. The classic way consists in its discretization and transformation on a system of linear equations whose coefficients are given by the mentioned proper and (in the main diagonal) improper integrals. Even when the improper integrals over particular elements (triangles or rectangles in 2D, tetrahedra or hexahedra in 3D) are believed to be calculable analytically (so that their values would be exact), the resultant formulae are extremely complicated and long. A better way is perhaps to carry out analytically only the first or (at most) the second integration while the last one can be performed by higher-order Gaussian quadrature with practically negligible error.

More sophisticated technique applicable for solution of the above system is the variational (Galerkin) approach with the possibility of using higher-order methods. This technique seems to be highly promising because it eliminates some of the above drawbacks. Necessary is much smaller number of elements in which the real distribution of the field quantity is approximated by higher-order polynomials. The total number of the degrees of freedom is substantially reduced and accuracy of results exponentially increases with the order of the considered polynomials. Careful selection of their coefficients also secures continuity of the approximated field quantity along the boundaries between particular elements in common with (when necessary) its derivatives. In association with suitable adaptivity techniques this methodology is supposed to become an extremely powerful tool for solution of specific eddy current problems. On the other hand, much more time and effort have to be exerted for preliminary computations of complicated doubled integrals representing elements of consequent numerical schemes. That is why no professional codes based on this technique are available so far and only few workshops deal with this topic.

The paper describes the integral approach and briefly discusses its individual steps. Two 2D examples calculated by own codes written by the authors illustrate its possible applications. The results are compared with values obtained analytically or from professional FEM-based codes.

2. FORMULATION OF THE PROBLEM

Consider a linear system consisting of free electrically conductive bodies $\Omega_i, i = 1,\dots,k$ and inductors $\Theta_i, i = k+1,\dots,n$ (see Fig. 1). The inductors are supposed to be supplied from sources of harmonic currents $I_{\mathrm{ext}\,i}, i = k+1,\dots,n$ (with corresponding current densities $\boldsymbol{J}_{\mathrm{ext}\,i}, i = k+1,\dots,n$) of pulsation ω. The system is supposed to operate in steady state and all elements can move at sufficiently low velocities (so that the component of eddy currents due to velocity may be neglected).

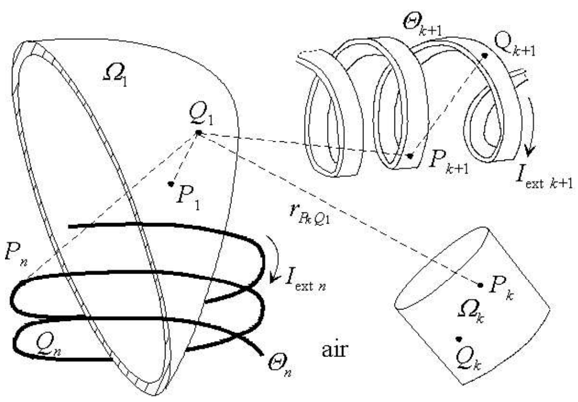

Fig. 1 Basic arrangement of the system

The task is to determine

- distribution of current densities within particular bodies of the system (in case that an inductor is wound by thin conductor, the skin effect in it can be neglected),
- distribution of the corresponding specific and total Joule losses,
- the Lorentz forces acting on individual elements.

Knowledge of these quantities is a must for eventual subsequent thermal (induction heating), mechanical and other computations.

3. MATHEMATICAL MODEL AND ITS SOLUTION

As all electromagnetic quantities are harmonic, we will express them in terms of their phasors. Phasor $\underline{\boldsymbol{A}}$ of the vector potential at any point $Q_j \in \Omega_j,\ j = 1,\dots,k$ is then given as a superposition of components produced by eddy current densities in free elements $\Omega_1,\dots,\Omega_k$ and components produced by total current densities in field coils $\Theta_{k+1},\dots,\Theta_n$. Now

$$\underline{\boldsymbol{A}}(Q_j) = \frac{\mu_0}{4\pi}\cdot\sum_{i=1}^{k}\int_{\Omega_i}\frac{\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i}(P_i)\cdot\mathrm{d}V}{r_{P_iQ_j}} + \frac{\mu_0}{4\pi}\cdot\sum_{i=k+1}^{n}\int_{\Theta_i}\frac{\underline{\boldsymbol{J}}_{\mathrm{tot}\,i}(P_i)\cdot\mathrm{d}V}{r_{P_iQ_j}}. \tag{1}$$

Here $\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i}(P_i),\ i = 1,\dots,k$ denotes the phasor of eddy current density at point $P_i \in \Omega_i$, $\underline{\boldsymbol{J}}_{\mathrm{tot}\,i}(P_i)$, $i = k+1,\dots,n$ the total external current density at point $P_i \in \Theta_i$ and $r_{P_iQ_j}$ is the distance between the reference point Q_j and general point of integration P_i (for an illustration, Fig. 1 depicts such a distance between points Q_1 and P_k).

The total current density $\underline{\boldsymbol{J}}_{\mathrm{tot}\,i}(P_i)$ in an inductor consists of the uniform current density $\underline{\boldsymbol{J}}_{\mathrm{ext}\,i}(P_i)$ delivered from the external current source and current density due to skin effect $\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i}(P_i)$ that is not known in advance. That is why equations analogous to (1) must generally be written even for the field coils. An exception holds for ideal thin conductors for which

$$\int_{\Theta_i}\frac{\underline{\boldsymbol{J}}_{\mathrm{tot}\,i}(P_i)\cdot\mathrm{d}V}{r_{P_iQ_j}} = \underline{I}_{\mathrm{ext}\,i}\cdot\int_{\Theta_i}\frac{\mathrm{d}\boldsymbol{l}(P_i)}{r_{P_iQ_j}} \tag{2}$$

and we can work directly with the field currents.

The Maxwell equation [4]

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t}(\mathrm{curl}\,\boldsymbol{A}) \tag{3}$$

generally yields solution

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \mathrm{grad}\,\varphi + \boldsymbol{g}(t) \tag{4}$$

where φ is any scalar function of the coordinates (that is usually interpreted as the electrical potential) and $\boldsymbol{g}$ any vector function of time. As no free body $\Omega_i,\ i = 1,\dots,k$ is supposed to be connected to the external voltage source and contains no other supplementary source of the electric field strength (for example, of the thermoelectric or photovoltaic origin), the second and third terms on the right-hand side of (4) vanish. The electric field strength in structures $\Omega_1,\dots,\Omega_k$ is, therefore, given only by the time variation of vector potential $\boldsymbol{A}$ and the phasor of eddy current density $\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i},\ i = 1,\dots,k$ can be obtained as

$$\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i} = \gamma_i \underline{\boldsymbol{E}}_i = -\mathrm{j}\cdot\omega\gamma_i\underline{\boldsymbol{A}} \tag{5}$$

where γ_i denotes the electrical conductivity of element Ω_i. After substituting (4) and (5) into (1) we get a system of integral equations for distribution of eddy current densities in structures $\Omega_1,\dots,\Omega_k$ in the form

$$\mathrm{j}\cdot\underline{\boldsymbol{J}}_{\mathrm{eddy}\,j}(Q_j) - \frac{\mu_0\omega\gamma_j}{4\pi}\cdot\sum_{i=1}^{k}\int_{\Omega_i}\frac{\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i}(P_i)\cdot\mathrm{d}V}{r_{P_iQ_j}} = \frac{\mu_0\omega\gamma_j}{4\pi}\cdot\sum_{i=k+1}^{n}\int_{\Theta_i}\frac{\underline{\boldsymbol{J}}_{\mathrm{tot}\,i}(P_i)\cdot\mathrm{d}V}{r_{P_iQ_j}},\quad j = 1,\dots,k. \tag{6}$$

Analogous equations may be derived even for field coils made from massive conductors. System (6) must be supplemented by boundary conditions that are mostly indirect and that evaluate the total current in the body. These conditions will be explained and discussed in more details in the next paragraphs.

The specific Joule losses w_{J} (whose distribution is necessary, for example, for consequent thermal calculations) at point $Q_j \in \Omega_j$, $j = 1,\ldots,k$ (in free bodies) are given by formula

$$w_{\mathrm{J}}\left(Q_j\right) = \frac{\left|\underline{\boldsymbol{J}}_{\mathrm{eddy}\,i}\left(Q_j\right)\right|^2}{\gamma_j} \tag{7}$$

while in field coils Θ_j, $j = k+1,\ldots,n$ (some of them may be massive)

$$w_{\mathrm{J}}\left(Q_j\right) = \frac{\left|\underline{\boldsymbol{J}}_{\mathrm{tot}\,j}\left(Q_j\right)\right|^2}{\gamma_j}. \tag{8}$$

The specific Lorentz forces $\boldsymbol{f}_{\mathrm{L}\,j}$ acting at point $Q_j \in \Omega_j$, $j = 1,\ldots,k$ may be calculated as

$$\boldsymbol{f}_{\mathrm{L}}\left(Q_j\right) = \boldsymbol{J}_{\mathrm{eddy}\,j}\left(Q_j\right) \times \boldsymbol{B}\left(Q_j\right). \tag{9}$$

It can be shown that this quantity can be transformed in terms of the corresponding phasors as

$$\begin{gathered}\boldsymbol{f}_{\mathrm{L}}\left(Q_j\right) = \underline{\boldsymbol{J}}_{\mathrm{eddy}\,j}\left(Q_j\right) \times \underline{\boldsymbol{B}}^*\left(Q_j\right) = \\ \underline{\boldsymbol{J}}_{\mathrm{eddy}\,j}\left(Q_j\right) \times \operatorname{curl} \underline{\boldsymbol{A}}^*\left(Q_j\right).\end{gathered} \tag{10}$$

The total Joule losses W_{J} in the $j-$th element or Lorentz force $\boldsymbol{F}_{\mathrm{L}}$ acting on it can be determined by integration of (8) or (10) over its volume V_j.

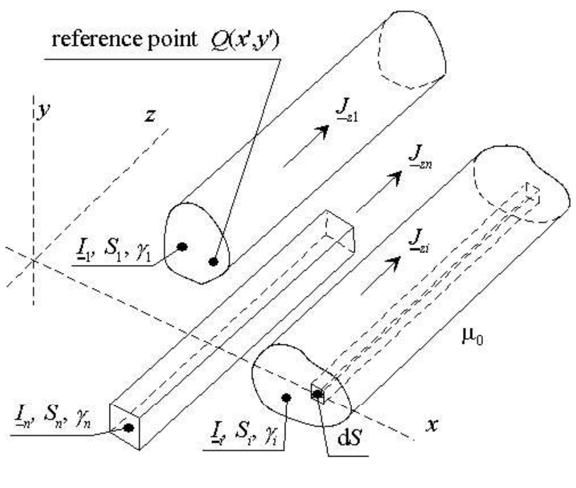

Fig. 2 General 2D arrangement

Solution to continuous model (6) can relatively easily be proved unambiguous [5]. But two serious problems have to be overcome to obtain particular results. The first of them is discretization of general 3D bodies and the second solution of the obtained large linear system characterized by a dense matrix. We were not able to develop and write a sufficiently general program so far. That is why we illustrate the methodology on simpler 2D arrangements.

Let us investigate a system of parallel massive conductors of any cross-section (see Fig. 2).

The conductors are nonmagnetic, their cross-sections do not change with length and they have the direction of axis z. They are supposed to carry harmonic currents of frequency f and amplitudes $\underline{I}_{\mathrm{ext}\,1},\ldots,\underline{I}_{\mathrm{ext}\,n}$. Current densities in them have only one nonzero component in direction z as well as the vector potential in the system.

The cross-sections of the conductors are denoted by symbols $S_1,\ldots,S_n$ and corresponding electrical conductivities $\gamma_1,\ldots,\gamma_n$.

The system (6) can now be transformed (details see [6], [7]). The phasor of total current density $\underline{J}_{\mathrm{tot}\,j}$ at a reference point $Q_j\left(x',y'\right)$ of the $j-$th conductor is given as

$$\begin{gathered}\underline{J}_{\mathrm{tot}\,j}\left(x',y'\right) = \underline{J}_{0j} + \\ +\sum_{i=1}^{n} \underline{k}_i \left[\int_{S_i} \underline{J}_{\mathrm{tot}\,i}\left(x,y\right) \ln\left(\left(x-x'\right)^2 + \left(y-y'\right)^2\right) \mathrm{d}S \right] \\ j = 1,\ldots,n\end{gathered} \tag{11}$$

where

$$\underline{k}_i = -\mathrm{j} \cdot \frac{\mu_0 \omega \gamma_i}{4\pi},\ \omega = 2\pi f \tag{12}$$

and $\underline{J}_{0j}$ is an unknown constant that has to be determined from indirect supplementary condition

$$\int_{S_j} \underline{J}_{\mathrm{tot}\,j} \mathrm{d}S = \underline{I}_j,\ j = 1,\ldots,n. \tag{13}$$

System (11) with conditions (13) may be solved numerically by the classical way or variational approach, as mentioned before. We briefly describe the principal steps of both algorithms.

Standard way of processing

The standard way of processing consists in discretization of the system and approximation of distribution of $\underline{J}_{\mathrm{tot}\,j}\left(x,y\right)$ in each element of each conductor by selected functions (that are mostly constant, linear or quadratic). We illustrate the method in more details using the approximation by constants.

Let the cross-section of the $j-$th conductor be discretized into m_j elements ΔS_{jk}, $k = 1,\ldots,m_j$ and in each element the physical distribution of the current density be approximated by constant $\underline{C}_{jk}$, $k = 1,\ldots,m_j$. Now we can rewrite (11) and (13) in the following way:

$$\underline{C}_{jl}\left(x_{jl},y_{jl}\right) = \underline{J}_{0j} + $$
$$+\sum_{i=1}^{n}\underline{k}_i\sum_{k=1}^{m_i}\underline{C}_{ik}\left[\int_{\Delta S_{ik}}\ln\left[\left(x-x_{jl}\right)^2+\left(y-y_{jl}\right)^2\right]\mathrm{d}S\right],$$
$$\sum_{k=1}^{m_j}\underline{C}_{jk}S_{jk} = \underline{I}_{\mathrm{ext}\,j},\ j=1,\ldots,n,\ l=1,\ldots,m_j \tag{14}$$

where point $x_{jl}, y_{jl} \equiv P_{jl}$ represent centre of gravity of element S_{jl} that is mostly of triangular or rectangular shape. Now it is necessary to determine integral

$$I_{ik} = \int_{\Delta S_{ik}}\ln\left[\left(x-x_{jl}\right)^2+\left(y-y_{jl}\right)^2\right]\mathrm{d}S \tag{15}$$

that may be either proper (when $P_{jl} \notin S_{ik}$) or improper ($P_{jl} \in S_{ik}$), $x, y \in \Delta S_{ik}$.

In case of *rectangular element* (Fig. 3) we have

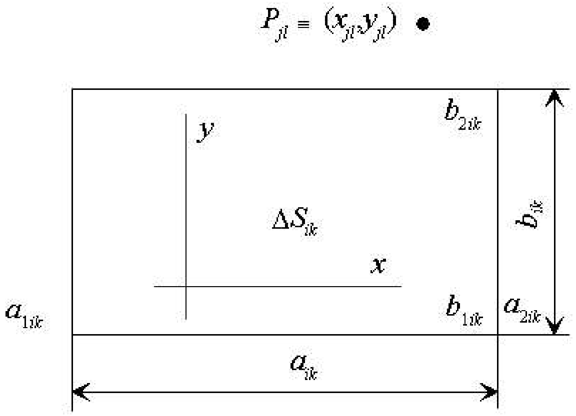

Fig. 3 To the calculation of I_{ik} over a rectangular element

$$\begin{aligned}Q_{ik} &= -A_{ik22} + A_{ik12} + A_{ik21} - A_{ik11} + \\ &+\left(B_{ik2}^2 - C_{ik2}^2\right)\cdot\mathrm{arctg}\frac{C_{ik2}}{B_{ik2}} + \\ &+\left(B_{ik1}^2 - C_{ik1}^2\right)\cdot\mathrm{arctg}\frac{C_{ik1}}{B_{ik1}} - \\ &-\left(B_{ik2}^2 - C_{ik1}^2\right)\cdot\mathrm{arctg}\frac{C_{ik1}}{B_{ik2}} - \\ &-\left(B_{ik1}^2 - C_{ik2}^2\right)\cdot\mathrm{arctg}\frac{C_{ik2}}{B_{ik1}} + \\ &+B_{ik2}C_{ik2}\ln\left[B_{ik2}^2 + C_{ik2}^2\right] + \\ &+B_{ik1}C_{ik1}\ln\left[B_{ik1}^2 + C_{ik1}^2\right] - \\ &-B_{ik2}C_{ik1}\ln\left[B_{ik2}^2 + C_{ik1}^2\right] - \\ &-B_{ik1}C_{ik2}\ln\left[B_{ik1}^2 + C_{ik2}^2\right]\end{aligned} \tag{16}$$

where

$$A_{ikpq} = 3a_{pik}b_{qik},\ B_{ikp} = x_{jl} - a_{pik},$$
$$C_{ikq} = y_{jl} - b_{qik},\ p,q = 1,2. \tag{17}$$

Computation of I_{ik} over a *general triangle* (see Fig. 4) is much more complicated. As the final formula takes a lot of place, we only indicate the way how to determine it. First we introduce transform

$$x = x_{1ik}u + x_{2ik}v + x_{3ik}\left(1-u-v\right),$$
$$y = y_{1ik}u + y_{2ik}v + y_{3ik}\left(1-u-v\right), \tag{18}$$
$$\mathrm{d}x\mathrm{d}y = 2\Delta S_{ik}\cdot\mathrm{d}u\mathrm{d}v,\ v\in\left\langle 0,1-u\right\rangle,\ u\in\left\langle 0,1\right\rangle$$

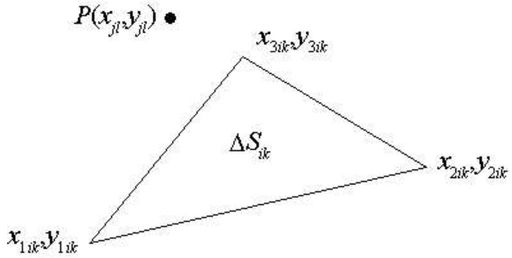

Fig. 4 To the calculation of I_{ik} over a triangular element

where ΔS_{ik} is the area of the triangle. Integral (15) may then be reformulated correspondingly and its evaluation can be realized, for example, by SW Mathematica. The more effective way is, however, to perform the first integral analytically and the second one by means of the Gauss quadrature formulas with practically negligible error.

More sophisticated could appear approximation of current density in the element by higher-order polynomial. In such a case, however, we have to select a reference point $P_{jl} \in \Delta S_{jl}$ for integral (15). But all points within the element can be declared equivalent, so that none of them should be preferred. This dilemma is, unfortunately, unsolvable in the indicated way, but by the following variational approach.

Variational approach

Variational approach allows using of higher-order approximation. Its first step consists in selection of suitable trial functions (usually polynomials) and approximation of the searched quantity in particular element by their combination. The next step is application of the Galerkin method in order to find the best coefficients of the combination representing the degrees of freedom.

Putting

$$\underline{J}_{ik} = \sum_{p}\underline{\alpha}_{pik}x^q y^r,\ q+r \le N \tag{19}$$

where N is the degree of the polynomials and $\underline{\alpha}_{pik}$ are its coefficients. Substituting (19) into (11) and (13) we obtain

$$\sum_p \underline{\alpha}_{pjl} x'^q y'^r = \underline{J}_{0zj} + $$
$$+\sum_{i=1}^{n} \underline{k}_i \sum_{k=1}^{m_i} \left[\int_{S_{ik}} \sum_p \underline{\alpha}_{pik} x^s y^t \ln\left[d^2\right] \mathrm{d}S \right], \qquad (20)$$
$$d^2 = (x-x')^2 + (y-y')^2 ,$$
$$j = 1,\ldots,n,\ \ s+t \le N,$$

$$\sum_{l=1}^{m_j} \int_{\Delta S_{jl}} \sum_p \underline{\alpha}_{pjl} x^q y^r \,\mathrm{d}S = \underline{I}_j ,\ \ j = 1,\ldots,n,\ \ q+r \le N . \qquad (21)$$

Application of the Galerkin technique now requires successive multiplication of (20) by particular terms of the polynomial in form

$$x'^{\sigma}\, y'^{\tau} \qquad (22)$$

and integration over element ΔS_{jl}. In this way we obtain a system of linear equations for coefficients $\underline{\alpha}$. Analytical computation of particular integrals representing the matrix coefficients is again possible for both rectangular and triangular elements; on the other hand, the resultant formulas are extremely complicated. A better way is perhaps to use Gauss' quadrature formulas, at least for the final integration.

4. ILLUSTRATIVE EXAMPLES

The *first example* is evaluation of movement (at velocity v) of a massive copper conductor of rectangular shape carrying current I of frequency f and phase shift φ over an electrically conductive thin copper plate (see Fig. 5). The solution is carried out by the classical method using constant approximation of current densities in particular elements.

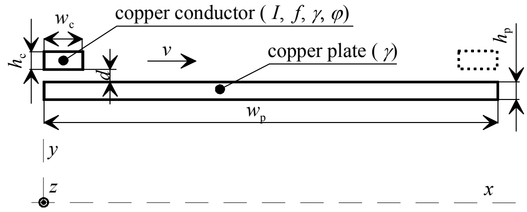

Fig. 5 Moving conductor over copper plate

Parameters:

- Plate - $w_p = 50$ mm, $h_p = 3$ mm, $\gamma = 57$ MS/m.
- Conductors - $w_c = 5$ mm, $h_c = 3$ mm, $\gamma = 57$ MS/m, $v = 2.7$ mm/s, $I = 150$ A, $f = 500$ Hz, $\varphi = 0$.
- Distance between the conductor and plate $d = 2$ mm.
- The system was supposed to be in harmonic steady state. At the starting time $t = 0$ the left side of the conductor was over the left side of the plate (see Fig. 5). Investigated were 10 positions of the conductor shifted successively by 5 mm, the last one being indicated by the dotted line in Fig. 5.

The example was calculated by own code (the discretization mesh had about 3600 elements) and the results were validated by FEM-based codes FEMM 4.0 (about 100000 elements) and QuickField 5.0 (130000 elements). The times of computation were quite comparable (each example took on a PC 2.6 GHz, 1GB RAM approximately 150 s). Such densities of meshes were proved sufficient to secure satisfactory geometrical convergence of the results. Some of these results are summarized and discussed in Table 1 and several following figures.

Tab. 1 contains important data concerning the position of the conductor and instantaneous values of the real and imaginary parts of the total current passing through it.

Tab. 1 Characteristic parameters of the conductor in particular positions

position of the left corner of the conductor (mm)	time (s)	phase shift (°)	Re(I) (A)	Im(I) (A)
0	0.00	0.00	150	0
5	1.85	57.76	-43.78	143.47
10	3.70	115.53	56.48	138.96
15	5.56	173.29	-68.71	-133.34
20	7.41	231.06	4.01	-149.95
25	9.26	288.82	-135.12	65.13
30	11.11	346.59	-48.97	-141.78
35	12.96	44.35	-123.03	-85.82
40	14.81	102.11	44.38	143.28
45	16.67	159.88	-122.31	86.84

Figs. 6 and 7 show the values of the corresponding Lorentz forces F_x and F_y acting on the conductor at its particular positions

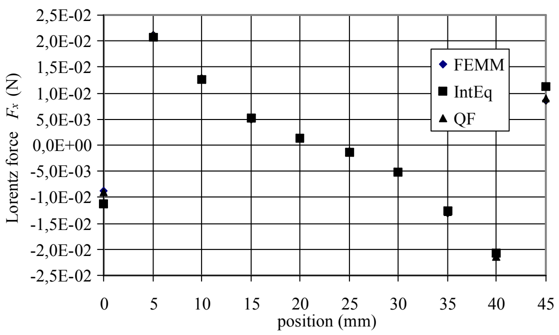

Fig. 6 Total horizontal force F_x acting on the conductor

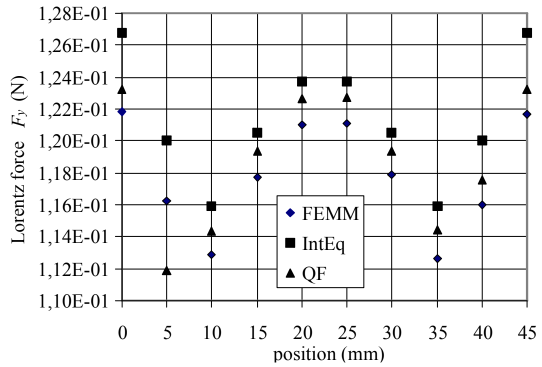

Fig. 7 Total vertical force F_y acting on the conductor

As can be seen from both figures, the maximum differences (especially in F_y) are about 10%. It is, unfortunately, not easy to decide what results are the best. Therefore, we prepared Fig. 8 that provides the total horizontal and vertical forces acting on the system (conductor and plate) that must identically be equal to zero.

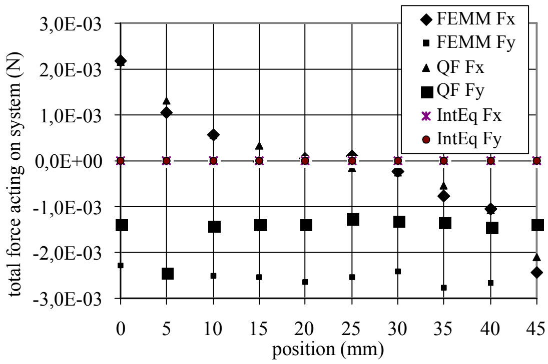

Fig. 8 Total forces acting on the system

While the integral method provides values of the order of 10^{-15} N, both FEM-based codes give values higher by about twelve orders. That is why the integral method is supposed to be more accurate in this case.

Figs. 9 and 10 show the values of the total average Joule losses in the conductor and in the plate

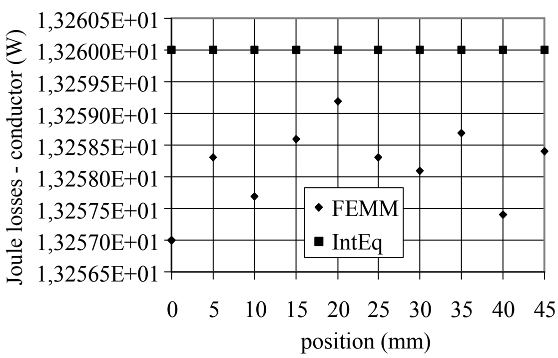

Fig. 9 Total average Joule losses in the conductor

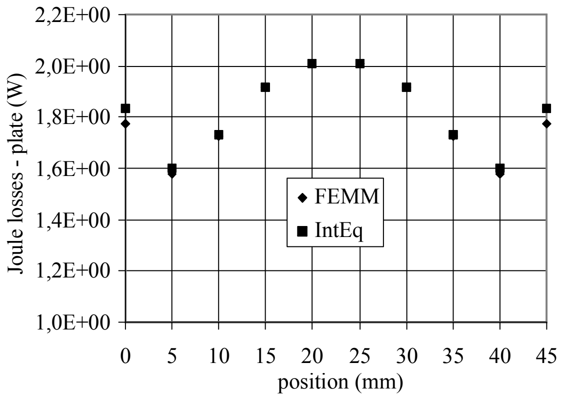

Fig. 10 Total average Joule losses in the plate

It can be seen (the results from QuickField are omitted) that even when the differences between both distributions are small, the more accurate values are provided again by the integral model (see Fig. 9).

Finally Fig. 12 shows distribution of the module of current density along vertical line going through the centre of the conductor and the plate for position 25 mm. Even here the results practically coincide.

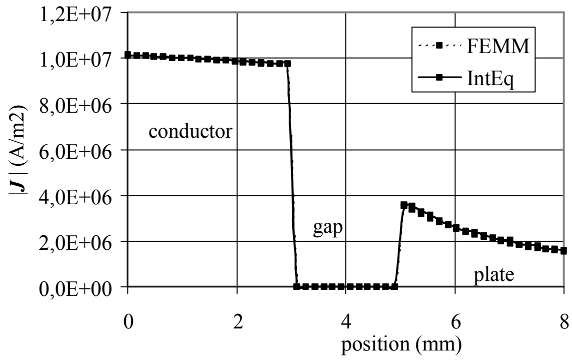

Fig. 11 Distribution of $|\boldsymbol{J}|$ along the vertical axis through the centre of conductor for position 25 mm

The *second problem* presents an application of the Galerkin method for computation of the skin-effect in a massive conductor of circular cross-section of radius R and electrical conductivity γ. The results are compared with values obtained analytically (well-known formulas containing the Bessel functions).

In [8] equation (11) was for a long cylindrical conductor transformed into

$$\underline{J}(a) = -\mathrm{j}\cdot\mu_0\omega\gamma\cdot\ln a\int_{r=0}^{a}\underline{J}(r)r\mathrm{d}r - \\ -\mathrm{j}\cdot\mu_0\omega\gamma\int_{r=a}^{R}\underline{J}(r)\ln r\cdot r\mathrm{d}r + \underline{J}_0 \qquad (23)$$

where a is the reference radius $0 \le a \le R$. This equation has naturally to be supplemented with condition (13).

Let us now suppose that

$$\underline{J}(r) = \sum_{i=1}^{n} \underline{\alpha}_i r^{i-1}. \quad (24)$$

After substituting (24) into (23) and consequent integration we get

$$\sum_{i=1}^{n} \underline{\alpha}_i a^{i-1} = \underline{J}_0 - -\mathrm{j} \cdot \mu_0 \omega \gamma \sum_{i=1}^{n} \underline{\alpha}_i \left[\frac{R^{i+1}}{i+1} \ln R - \frac{R^{i+1} - a^{i+1}}{(i+1)^2} \right]. \quad (25)$$

This equation has to be successively multiplied by a^{j-1}, $j = 1,\ldots,n$, and integrated from 0 to R. In this way we obtain a system of equation in the form

$$\sum_{i=1}^{n} \underline{\alpha}_i R^{i-1} \left[\frac{\underline{k}R^2}{i+1} \cdot \ln R - \frac{\underline{k}R^2}{(i+1)(i+j+1)} - \frac{j}{i+j-1} \right] + +\underline{J}_z = 0,\ j = 1,\ldots,n \quad (26)$$

where $\underline{k} = -\mathrm{j} \cdot \mu_0 \omega \gamma$. Condition (13) may be modified as

$$\sum_{i=1}^{n} \underline{\alpha}_i \frac{R^{i+1}}{i+1} = \frac{\underline{I}}{2\pi}. \quad (27)$$

The matrix of system consisting of (26) and (27) is, unfortunately, ill-conditioned, so that its inverse has to be performed extremely carefully.

Table 1 Real parts of the total current density

radius r (mm)	Re[**J**] (A/mm^2) exact analytical solution	Re[**J**] (A/mm^2) numerical integral solution (order 0) 10 rings	Re[**J**] (A/mm^2) numerical integral solution Galerkin (order 5) 1 ring	Re[**J**] (A/mm^2) numerical integral solution Galerkin (order 10) 1 ring
0.5	-0.096157	-0.096473	-0.095760	-0.096157
1.5	-0.082941	-0.082955	-0.079111	-0.082941
2.5	-0.054649	-0.054602	-0.055180	-0.054649
3.5	-0.008025	-0.006610	-0.011517	-0.008025
4.5	0.060590	0.062921	0.058405	0.060590
5.5	0.153608	0.156704	0.155191	0.153608
6.5	0.269773	0.273107	0.273523	0.269773
7.5	0.400899	0.403433	0.402164	0.400899
8.5	0.527887	0.527976	0.523950	0.527887
9.5	0.616437	0.611863	0.615800	0.616437

An example has been calculated with parameters $R = 10$ mm, $I = 100$ A, $f = 500$ Hz and electrical conductivity of used material $\gamma = 59$ MS/m. The results calculated by a procedure written in the environment of SW Mathematica are summarized in Tabs. 1 and 2. The second column contains analytically obtained value, the third column results calculated numerically by standard zero-order numerical solution of, the fourth column by the Galerkin approach for $n = 5$), and the last column by the same method for $n = 10$. The last results are in accordance with the analytically obtained values practically in six valid digits.

Table 2 Imaginary parts of the total current density

radius r (mm)	Im[**J**] (A/mm^2) exact analytical solution	Im[**J**] (A/mm^2) numerical integral solution (order 0) 10 rings	Im[**J**] (A/mm^2) numerical integral solution Galerkin (order 5) 1 ring	Im[**J**] (A/mm^2) numerical integral solution Galerkin (order 10) 1 ring
0.5	0.111400	0.113688	0.112112	0.111400
1.5	0.122314	0.124578	0.121335	0.122314
2.5	0.141794	0.143922	0.141234	0.141794
3.5	0.164794	0.166556	0.165281	0.164794
4.5	0.182963	0.183965	0.183835	0.182963
5.5	0.183981	0.183678	0.184135	0.183981
6.5	0.151197	0.148956	0.150309	0.151197
7.5	0.064038	0.059324	0.063366	0.064038
8.5	-0.100109	-0.107436	-0.098799	-0.100109
9.5	-0.361779	-0.370997	-0.361406	-0.361779

5. CONCLUSION

Integral analysis of eddy currents and other associated electromagnetic quantities in linear systems represents a reliable and powerful tool providing direct distribution of current density without necessity of evaluating the field quantities. The only serious problem is solution of large fully populated matrices that follow from discretization of the continuous mathematical model. This problem can be, however, avoided (at least to some extent) by using higher-order Galerkin approach that seems to provide extremely accurate approximations.

Of course, higher accuracy of results is paid for by much more complicated computation of coefficients of the system matrix. These are given by doubled integrals over various geometrical elements, whose determination is extremely laborious. Nowadays, we are enumerating these integrals over rectangular or hexahedral elements, which is possible (even when with considerable complications) in the analytical way. Much more complicated is their computation over general triangles (2D) or tertahedra (3D). Even when some of them (but perhaps only for low degrees of approximation polynomials) may also be evaluated analytically, the more promising way seems to be using the Gauss quadrature in 2D or 3D.

Planned is further extensive investigation in the field aimed at mathematical aspects (velocity, accuracy) of the method and possibilities of its practical application. The results will be validated by comparison with other reliable methods and with experiments.

The methodology is planned to be used for solution of a lot of problems concerning eddy currents, for example induction heating, processing of nonferromagnetic molten metals, electrodynamic levitation and many others.

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BIOGRAPHY

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