ALTERNATIVE APPROACH TO DATA NETWORK OPTIMIZATION

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SUMMARY

This paper offers another than standard view of solving the optimization problems occurring in practice. For description of a given problem we will take a suitable mathematical model which uses the graph theory and the theory of computing complexity. The introduction explains basic terms and it contains the most widely known standard combinatorial problems on graphs. The next chapter describes a particular practical problem. The problem is to find an optimal distribution of the data network among single offices and components of self government when we consider more providers of telecommunication connections and various technical conditions of data transmission. We will transform this problem into one correspondent mathematical model. In this mathematical model, the graph and the used objective function will represent the given network with all its conditions, requirements and also with its limitations. We will find the solution of this optimization problem by solving this mathematical model through the use of known algorithms. The goal of this paper is to show that this alternative approach to the given problem offers more simple and more elegant solution in comparison to solutions of some practical problems using standard optimization methods which can be slow and complicated.

Keywords: *spanning tree, objective function, optimization problem, algorithm, mathematical model*

1. INTRODUCTION

It is not satisfactory anymore to transform the optimization problems, which rise from technological progress and from the development of IT, into the standard optimization problems, which have mainly been used in their solvings. We also have to seek new mathematical models, which will enable us to give a look at the other optimization alternatives as well as at the new forms of evaluation and determination. Usage of graph theory and theory of computing complexity is a very suitable mathematical model for discreet structures and actions.

Combinatorial optimization problem **P** is a pair **P** = (**D**, f), where **D** is a finite set of all feasible solutions (feasible sets) of a query **P** and f is an objective function. There are two separate basic versions of the problem in the theory of computing complexity.

1. If we seek such feasible set D from the system of feasible sets **D**, so that the value of objective function $f(D)$ was minimal (maximal) on **D**, we call it optimization version of the problem.
2. If we want to decide, if there is such feasible set D in the system of feasible sets **D**, so that unequality $f(D) \leq k$, ($f(D) \geq k$) applies for predetermined k, we call it decision version of the problem.

Sum, bottleneck and balanced problems belong to the best known standard combinatorial problems on graphs. We can formulate them as follows:

Let us have the graph $G = (V, E)$, where V is set of vertices, E is set of edges and the system of the feasible sets $\mathbf{D}(G)$. Let us assign the weight $w(e) \in \langle 0, \infty)$ to each edge $e \in E$ and let us have standard objective functions:[1]

Sum function:

$$f_1 = \sum_{e \in D} w(e), \text{ where } D \in \mathbf{D}(G),$$

Bottleneck function:

$$f_2 = \max_{e \in D} w(e), \text{ where } D \in \mathbf{D}(G),$$

Balanced function:

$$f_3 = \max_{e \in D} w(e) - \min_{e \in D} w(e), \text{ where } D \in \mathbf{D}(G).$$

Then we get following types of optimization problems:

Sum problem: $$f_1(D) \xrightarrow[D \in \mathbf{D}(G)]{} \min,$$

Bottleneck problem: $$f_2(D) \xrightarrow[D \in \mathbf{D}(G)]{} \min,$$

Balanced problem: $$f_3(D) \xrightarrow[D \in \mathbf{D}(G)]{} \min.$$

2. PROBLEM

At the present time, the effective communication among single components of self government, eventually among offices of public administration is increasingly important by the extensive competence transfer to the self governments. The communication networks serve those purposes (data networks are the most suitable ones). The single communication connections are operated by different providers under different economic and technical conditions. Let us have a group of telecommunication connection providers for a given telecommunication area that is represented by the communication network. Each direct connection between two junctions is realized by exactly one provider. Each connection could be realized for a prearranged

[1] Those are most widely used objective functions in mathematical optimisation.

charge, and the transmission capacity of single connections per unit of time are known. The prearranged charge will guarantee the minimal transmission capacity per unit of time, which we will use by a given provider and the data transmission over this limit will be charged by the particular provider according to separate tariff rate. We also know maximum possible transmission capacities for each connection.

From the standard economic point of view we would like to minimize the data transmission expenses in order to ensure the connection to each office, so that none of the offices will remain isolated.

We will solve the problem of such connection distribution for single offices in this paper, that approximately same dataflow will be provided with the lower-bound requirement for the data transmission capacity per unit of time among single offices. This approach is justifiable. By using the first optimization approach (strictly financial point of view) the following situation could occur: the problem with information availability from some offices can uprise at the price of lowest expenses or the other way round, which means that the low data transmission capacity per unit of time can cause information inapplicability in real time, that will cause malfunction or inefficiency of the particular network.

3. MATHEMATICAL MODEL

Let us repeat that the combinatorial optimization problem **P** is a pair (**D**, f), where **D** is a finite set of all feasible solutions of the given problem **P** .

$$\mathbf{D} = \{D;\ D \text{ is feasible solution of the problem } \mathbf{P}\}$$

and f is an objective function. The task is to find the optimal feasible solution D from the system of sets **D**. We are dealing with the minimization of the objective function f in this paper, and therefore we can write the problem **P** in the form:

$$f(D) \xrightarrow[D \in D]{} \min .$$

We are dealing with optimization task on graphs where the feasible sets are the subsets of the set of edges of the given graph – subgraphs. We want to get a minimal connected subgraph containing all the vertices which is the spanning tree of the graph. This problem was formulated in [6], [7].

Let us have the graph $G = (V, E)$ and the decomposition of the edge set E into disjoint categories $S_1, \ldots, S_p$; it is ensured this way, that each edge of the spanning tree T belongs exactly to one of these categories. For each edge $e \in E$ is defined the weight $w(e)$, where $w(e)$ is nonnegative integer and our objective function f has the form $L(T)$:

$$L(T) = \max_{1 \le i \le p} \left(\max_{e \in S_i \cap T} w(e) - \min_{e \in S_i \cap T} w(e) \right),$$

where $T \in \mathbf{D}(G)$ and we assume that $\max_{e \in \emptyset} w(e) = 0$. (In this case $\mathbf{D}(G)$ presents the set of all spanning trees, but generally $\mathbf{D}(G)$ can also present the ways between vertices a and b, or perfect pairings in graph G.)

Let us note that the problems with the above defined objective function L come down to the standard balanced problem, if the number of categories $p = 1$, which means, that we consider such case, where all the connections are covered by one provider. If the number of categories p equals to the number of edges (every possible connection is covered by a different provider), then the problem with the objective function loses the significance, because it comes down to the problem with constant objective function equal 0.

The first remarks about complexity of similar problems appeared in authors Averbach, Berman [3], Richey and Punnen [14] and Punnen [13]. We can also find the problems with a similar idea of edges categorization in the papers [1, 2, 4, 5, 6, 8, 9, 10, 12]. Mathematical terms and basic facts related with the graph theory which are discussed in this paper can be found for instance in authors Chartrand, Oellermann [11].

Let us retransform the problem from the chapter 2 into a mathematical model defined in this chapter. The simple graphical illustration of the transformation of the given problem is visible on the Figures 1-6.

Let us assume, that there are p data services providers available. The regional data network will be represented by the graph $G = (V, E)$ (Fig. 1), where the set of vertices V of the graph G presents the data services providers and their customers.

We will not consider an unconnected graph. Such graph would represent a data network where some of the providers or customers or a group of providers and customers cannot be connected with the other providers. This way defined problem would not have any solution.

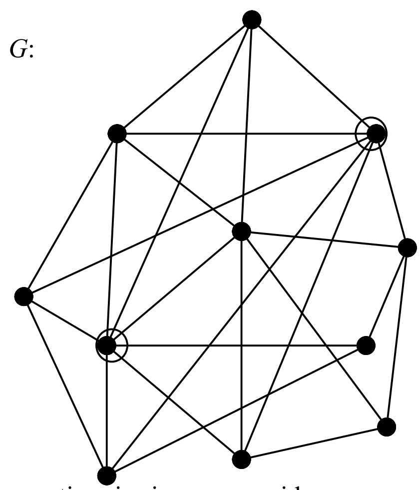
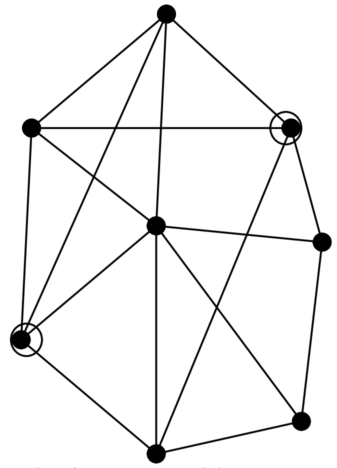


Fig. 1 G: graph presenting the connections among all the customers and the providers – general data network

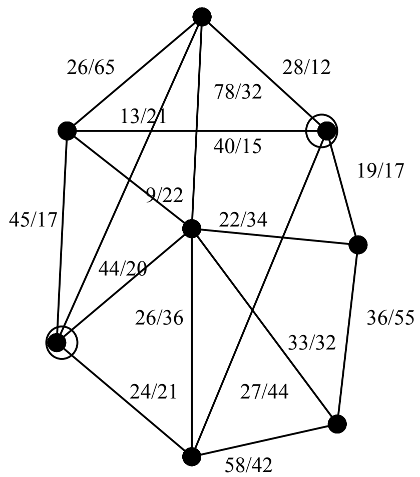
U:



- vertices in ring are providers.

Fig. 2 *U*: graph presenting the connections among the offices and the providers

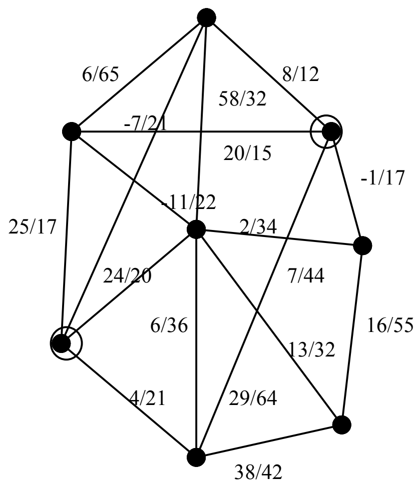
Ũ:



- vertices in ring are providers.

Fig. 3 *Ũ*: graph presenting the connections among the offices and the providers with weights of edges

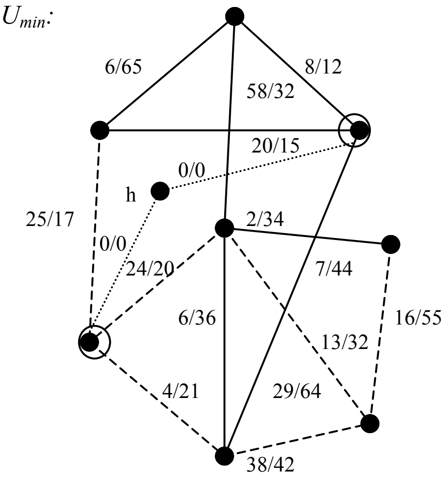
Ū:



lower bound of capacity = 20
- vertices in ring are providers.

Fig. 4 Ū: modified graph presenting the connections among the offices and the providers with respect to the lower bound of capacity

U_{min}:

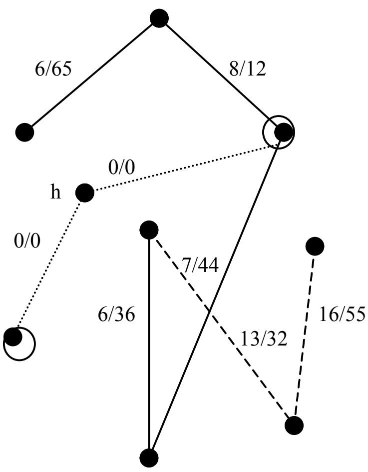


- vertices in ring are providers.

edges from category S_0
edges from category S_1
edges from category S_2

Fig. 5 U_{min}: resultant graph describing the problem P

T^{opt}:



- vertices in ring are providers.

Fig. 6 T^{opt}: resultant graph – spanning tree describing the problem P. $L(T^{opt}) = \max\{0;2;3\} = 3$

The set of edges E is created by the edges that couple a pair of graph vertices if and only if there is a data connection between them.

As we are interested only in offices of public and state administration, we will create an induced subgraph U (Fig. 2) on the set of vertices $V(U)$ of the graph G, where $V(U)$ is the set of those vertices of the graph G, which match the offices of public and state administration and the data services providers.

Graph $\tilde{U}$ is graph with weights of edges. We will assign to the edges a pair of weights, where the first weigh presents the data transmission capacity per unit of time between the pair of vertices and the second weight defines the expenses for utilization of services provider on the particular edge (Fig. 3).

We have to adapt the weights of edges in this subgrap as well, because we need to guarantee the condition of the lower boundary of the transmission capacity per unit of time. We can reach that by modifying the first weights of edges so that we will reduce them by the value of the lower bound (graph $\bar{U}$ on the Fig. 4).

Then we will also modify the subgraph so that we eliminate all the edges which have negative first weight because those connections will not guarantee the wanted transmission capacity, therefore we cannot use them. We will mark this subgraph as $U_{\min}$. We will divide the edges in the subgraph $U_{\min}$ into p categories so that the edges, which ensure the data transmission for the same providers, we will put into the same category. We will get p categories of edges S_1, ... , S_p. So we could guarantee the option that any of service providers can be eliminated from the selection, we will create one special vertex h, which we will bind by an edge with each vertex, which represents the data services provider (Fig. 5). We will assign to those edges the first and the second weight equal 0 and we will add all these edges to the new created zero category S_0. This way is the graph construction finished.

The problem **P** is given by the graph $U_{\min}$, by the decomposition of the edges into the categories and by the objective function L. The system of the feasible sets **D** contains all spanning trees T of the graph $U_{\min}$. If there was no spanning tree in the graph, there would be an isolated office or the set of offices, which would have no connection with the other ones. The aim is to find such spanning tree T from the set **D**, for which the value of the objective function L will be minimal with respect to the first weights. This means that there will be found an edge with maximal provided transmission capacity ($\max_{e \in S_i \cap T} w(e)$) and an edge with minimal provided transmission capacity ($\min_{e \in S_i \cap T} w(e)$) for each provider in the given spanning tree. We determine their differences and after that we determine the maximal difference of all of them

$$(\max_{1 \le i \le p} \left(\max_{e \in S_i \cap T} w(e) - \min_{e \in S_i \cap T} w(e) \right)).$$

The spanning tree with the minimal value of the maximal differences is the optimal solution T^{opt} (Fig.6). That presents the transmission network with the smallest differences in the amount of data transmitted per unit of time among the offices. We will obtain this way the balance of the transmission and this also implies the balance of the expenses for single offices. The complexity of general case of problem **P** was published in [6] and its polynomial algorithm for solving of the problem **P** was published in [7].

Our search for optimum solutions of some problems we will extensively use the following test.

Feasibility test:

For a given problem of the form

$$L(T) \xrightarrow[T \in \mathrm{D}]{} \min$$

and a set $E_0 \subseteq E$ (E_0=allowed elements) decide, whether there exists a feasible solution $T \in \mathbf{D}(G)$, containing only elements of set E_0.

In cases when $\mathbf{D}(G)$ consists of spanning trees, perfect matchings or $a-b$ path problems there are well known polynomial algorithms for solving the feasibility test.

We briefly show that all the considered problems with objective function L is polynomially solvable for p fixed, $p \ge 2$.

Denote by

$[\alpha,\ \beta]_T$ for $T \subseteq E$ the set $\{e \in T;\ \alpha \le w(e) \le \beta\}$.

Advance for finding of optimum of problem L-spanning tree.

For each category S_i and each feasible solution T let $\alpha_i = \min_{e \in S_i \cap T} w(e)$ and $\beta_i = \max_{e \in S_i \cap T} w(e)$.

Then T is a subset of union $\bigcup_{i=1}^{p} M_i$, where $M_i = [\alpha_i, \beta_i]_{S_i}$ and $L(T) = \max_{1 \le i \le p}(\beta_i - \alpha_i)$.

It therefore suffices to check all possible unions (using all combinations of p-tuples of α_i, β_i pairs) whether where exists a feasible solution $T \subseteq \bigcup_{i=1}^{p} M_i$.

Here we can employ the feasibility test.

Clearly the minimum value of $L\left(\bigcup_{i=1}^{p} M_i\right)$ taken over union $\bigcup_{i=1}^{p} M_i$ containing a feasible solution is the optimum of problem $L(T) \xrightarrow[T \in \mathrm{D}]{} \min$.

Since for each M_i there are at most $|E|^2$ possibilities for choosing upper and lower bound α_i, β_i it sums up for all categories to $|E|^{2p}$ times performing the feasibility test. This takes only polynomial time providet that a polynomial feasibility test exists.

If this previous way [7] will by applicated we find T^{opt} - optimal solution of problem L-spanning tree, if solution exists.

4. CONCLUSION

In this paper we have shown the specific option how to apply the mathematical optimization model using graph theory. It is possible to use similar approach not only for our problem solving from the chapter 2. A similar problem can be use also for various technical or industrial applications. There are known many other objective functions except for the mentioned objective function which are published [6] and which present the problem optimization from various point of view and which consider different characteristics of graphs. The algorithms for solutions of these problems are known [7]. For instance, if we were interested exclusively in expenses minimization in our problem, it would be the standard problem of minimal spanning tree, solution of which is generally known. Mathematical modelling of similar problems optimization finds a large versatility in praxis.

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