INTERVAL EIGENVECTORS OF CIRCULANT MATRICES IN FUZZY ALGEBRA

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ABSTRACT

Fuzzy algebra is an algebraic structure in which classical addition and multiplication are replaced by $\oplus$ *and* $\otimes$*, where* $a \oplus b = \max\{a,b\}$, $a \otimes b = \min\{a,b\}$. *A vector x is an eigenvector of a matrix A if* $A \otimes x = x$.

An interval vector $\mathbf{X}$ *and the possible and universal eigenvectors are defined. A necessary and sufficient condition for the possible and universal eigenvectors of a circulant matrix are proved and several examples are given.*

Keywords: *fuzzy algebra, circulant matrix, interval vector, possible eigenvector, universal eigenvector*

1. INTRODUCTION

Matrices in fuzzy algebra are useful for expressing applications of fuzzy discrete dynamic systems, graph theory, scheduling, medical diagnosis [13], [14] or fuzzy logic programs [7].

Eigenvector of a fuzzy matrix characterize stable states of the corresponding discrete event systems. Investigation of the fuzzy eigenvectors of a given matrix is therefore of great importance. The eigenproblem in fuzzy algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace and the algorithms for computing the maximal eigenvector of a given matrix were suggested, see, e.g., [1], [10], [11], [15]. The structure of the eigenspace as a union of intervals of increasing eigenvectors is described in [3].

The eigenproblem of fuzzy matrices and its connection to paths in digraphs were investigated in [1], [6].

The structure of the eigenspace for a special case of so-called circulant matrices is described in [5].

In practice, vector inputs are rather contained in some intervals than exact values. The aim of this paper is to describe the interval eigenvectors of circulant matrices. We define the possible and universal eigenvectors of circulant matrices and give necessary and sufficient conditions for them.

2. PRELIMINARIES

The fuzzy algebra $\mathscr{B}$ is the triple $(B,\oplus,\otimes)$, where $(B,\leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$ and $\otimes$, respectively. The least element in B will be denoted by O, the greatest one by I.

By $\mathbb{N}$ we denote the set of all natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$. For a given natural number $n \in \mathbb{N}$, we use the notations $N = \{1,2,\ldots,n\}$ and $N_0 = \{0,1,\ldots,n-1\}$.

For any $n \in \mathbb{N}$, $B(n,n)$ denotes the set of all square matrices of order n and $B(n)$ the set of all n-dimensional column vectors over B. The matrix operations over $\mathscr{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field.

For a given matrix $A \in B(n,n)$, the number $\lambda \in B$ and the n-tuple $x \in B(n)$ are the so-called *eigenvalue* and *eigenvector* of A, respectively, if

$$A \otimes x = \lambda \otimes x. \tag{1}$$

For $\lambda = I$ equation (1) gets the form

$$A \otimes x = x. \tag{2}$$

The *eigenspace* $V(A)$ is defined as the set of all eigenvectors of A, i.e.,

$$V(A) = \{x \in B(n);\ A \otimes x = x\}.$$

3. EIGENVECTORS OF CIRCULANT MATRICES

A square matrix is circulant if the input values in every row are the same as the values in the previous row, but they are cyclically shifted by one position to the right. Formally, matrix $A \in B(n,n)$ is circulant if

$$a_{ij} = a_{kl}$$

whenever

$$i-k \equiv j-l \mod n.$$

Hence, a circulant matrix A is completely determined by inputs $a_0, a_1, \ldots, a_{n-1}$ in the first row, i.e.,

$$A(a_0,a_1,\ldots,a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_0 \end{pmatrix}.$$

For a given circulant matrix $A(a_0,a_1,\ldots,a_{n-1})$ we define a strictly decreasing sequence $M(A) = (m_1, m_2, \ldots m_{s(A)})$ recursively as follows:

$$m_r = \begin{cases} \max\{a_i;\ i \in N_0\} & \text{for } r = 1, \\ \max\{a_i;\ a_i < m_{r-1};\ i \in N_0\} & \text{for } r > 1. \end{cases}$$

It is easy to see that $m_1 > m_2 > \cdots > m_{s(A)}$ and the length $s(A)$ of the sequence $M(A)$ equals to the number of different values in the first row. We shall use the notation $S(A) = \{1,2,\ldots,s(A)\}$. For each $r \in S(A)$ denote by P_r the set of all positions of the value m_r in first row, i.e.,

$$P_r = \{i \in N_0;\ a_i = m_r\}$$

and define the numbers

$$d_r = \gcd(P_r \cup \{n\}), \;\; e_r = \gcd(d_1, d_2, \ldots, d_r) = \gcd(e_{r-1}, d_r). \tag{3}$$

The following two lemmas give necessary conditions for a vector x to be an eigenvector of A.

Lemma 3.1. [5] *Let circulant matrix $A = A(a_0, a_1, \ldots, a_{n-1})$ be given and x be an eigenvector of A. Then $x_k \leq m_1$ for every $k \in N$.*

Lemma 3.2. [5] *Let circulant matrix $A = A(a_0, a_1, \ldots, a_{n-1})$ be given and x be an eigenvector of A, let $k, l \in N$ and $r \in S(A)$. If $x_k < m_r$ then the following implications hold true*

(i) if $k \equiv l \bmod d_r$ then $x_k = x_l$,

(ii) if $k \equiv l \bmod e_r$ then $x_k = x_l$.

In fact, in the previous lemma the part (ii) implies the part (i). The lemma says that in each eigenvector any entry $x_k < m_r$ must be repeated after e_r positions. The following theorem gives a necessary and sufficient condition for a vector x to be an eigenvector of A.

Theorem 3.1. [5] *Let $A = A(a_0, a_1, \ldots, a_{n-1})$ be a circulant matrix. A vector $x \in B(n)$ is an eigenvector of A if and only if there is a partition $\mathscr{T}$ on N such that for every class $T \in \mathscr{T}$ there exist $x(T) \in B$ and $r(T) \in S(A)$ satisfying the following conditions*

(i) $x_k = x(T) \leq m_1$ for every $k \in T$,

(ii) $r(T) = \max\{r \in S(A); x(T) < m_r\}$,

(iii) T is an equivalence class in N modulo $e_{r(T)}$.

For a given $r \in S(A)$ denote $E^r = \{1, 2, \ldots, e_r\}$. For each $r \in S(A)$, $i \in E^r$ define

$$N_i^r = \{k \in N; k \equiv i \bmod e_r\}, \tag{4}$$

i.e., N_i^r is the equivalence class modulo e_r containing $i \in E^r$.

Remark 3.1. *Since $e_{r+1} | e_r$ for each $r \in \{1, 2, \ldots, s(A)-1\}$, it follows from the properties of the equivalence relation that for each $i, j \in N$ and $r_1, r_2 \in S(A)$, $r_1 \leq r_2$ either $N_i^{r_1} \subseteq N_j^{r_2}$ or $N_i^{r_1} \cap N_j^{r_2} = \emptyset$.*

The following theorem gives other necessary and sufficient condition for a vector x to be an eigenvector of A, which will be used in the next section.

Theorem 3.2. *Let $A = A(a_0, a_1, \ldots, a_{n-1})$ be a circulant matrix. A vector $x \in B(n)$ is an eigenvector of A if and only if $\max_{k \in N} x_k \leq m_1$ and for each $r \in S(A)$, $i \in E^r$ such that $\min_{k \in N_i^r} x_k < m_r$ the equality $x_t = x_s$ is satisfied for each $t, s \in N_i^r$.*

Proof. Suppose that

$$\max_{k \in N} x_k \leq m_1 \tag{5}$$

and

$$(\forall r \in S(A))(\forall i \in E^r)[\min_{k \in N_i^r} x_k < m_r \Rightarrow (\forall t, s \in N_i^r) x_t = x_s]. \tag{6}$$

We construct a partition $\mathscr{T}$ on N satisfying the conditions from Theorem 3.1 by the following algorithm.

Algorithm $\mathscr{A}$: Determining a partition $\mathscr{T}$

Input: $A(a_0, a_1, \ldots, a_{n-1})$, vector $x \in B(n)$
Output: the sets T^j, the elements $x(T^j) \in B$ and the numbers $r(T^j) \in S(A)$

begin
  $j := 1; \tilde{N} := \emptyset;$
  for $r = s(A) : 1$ **do**
    if $\tilde{N} \neq N$ **then**
      for $i = 1 : e_r$ **do**
        if $\min_{k \in N_i^r} x_k < m_r$ and $i \notin \tilde{N}$ **then**
          $T^j := N_i^r; r(T^j) := r; x(T^j) := \min_{k \in T^j} x_k;$
          $\tilde{N} := \tilde{N} \cup T^j; \; j := j + 1;$
        endif
      enddo
    endif
  enddo
end

As the output of Algorithm $\mathscr{A}$ we get the sets $T^1, T^2, \ldots, T^p$, $1 \leq p \leq n$. We will prove that the sets $T^1, T^2, \ldots, T^p$, $1 \leq p \leq n$ create the partition $\mathscr{T}$ on N. The inequality $T^u \neq T^v$ for $u \neq v$ follows from Remark 3.1. Further, for each $l \in N$ there exists $j \in S(A)$ such that $\min_{k \in N_i^j} x_k < m_j$, where $l \in N_i^j$, i.e., $l \in T^s$ for some $s \leq n$, which implies $\bigcup_{k=1,\ldots,p} T^k = N$.

The validity of the conditions $ii)$ and $iii)$ of Theorem 3.1 follows from the definitions of T^j and $r(T^j)$ for each $T^j \in \mathscr{T}$.

The equality $x(T^j) = x_k$ for each $k \in T^j$ follows from (6) and the inequality in $i)$ is a consequence of (5). Thus the condition $i)$ is satisfied for each $T^j \in \mathscr{T}$.

By Theorem 3.1 the vector x is an eigenvector of A.

For the converse implication suppose that

$$\max_{k \in N} x_k > m_1 \tag{7}$$

or

$$(\exists r \in S(A))(\exists i \in E^r)[\min_{k \in N_i^r} x_k < m_r \wedge (\exists t, s \in N_i^r) x_t \neq x_s]. \tag{8}$$

If inequality (7) is satisfied then x is not an eigenvector of A by Lemma 3.1.

If condition (8) holds true then there exists $r \in S(A)$, $i \in E^r$ and $l, m \in N_i^r$ such that $x_l = \min_{k \in N_i^r} x_k < m_r$ and $x_m \neq x_l$.

Since $l \equiv m \bmod e_r$, by Lemma 3.2ii) the vector x is not an eigenvector of A. $\square$

The eigenproblem for circulant matrices in max-plus algebra was studied in [10].

4. INTERVAL EIGENVECTORS

Similarly to [2], [4], [8], [9] we define an interval vector $\mathbf{X}$.

Definition 4.1. *Let* $\underline{x}, \overline{x} \in B(n)$, $\underline{x} \leq \overline{x}$. *An interval vector* $\mathbf{X}$ *with bounds* $\underline{x}, \overline{x}$ *is defined as follows*

$$\mathbf{X} = [\underline{x}, \overline{x}] = \{x \in B(n); \underline{x} \leq x \leq \overline{x}\}.$$

Definition 4.2. *An interval vector* $\mathbf{X}$ *is*

(i) a possible eigenvector *of* A *if there exists* $x \in \mathbf{X}$ *that is an eigenvector of* A*,*

(ii) a universal eigenvector *of* A *if each vector* $x \in \mathbf{X}$ *is an eigenvector of* A*.*

4.1. Possible eigenvectors

Theorem 4.1. *Let* $A = A(a_0, a_1, \ldots, a_{n-1})$ *be a circulant matrix. An interval vector* $\mathbf{X}$ *is a possible eigenvector of* A *if and only if* $\max_{k \in N} \underline{x}_k \leq m_1$ *and for each* $r \in S(A)$, $i \in E^r$ *such that* $\min_{k \in N_i^r} \overline{x}_k < m_r$*, the inequality* $\max_{k \in N_i^r} \underline{x}_k \leq \min_{N_i^r} \overline{x}_k$ *holds true.*

Proof. In formally way, the theorem says that $\mathbf{X}$ is a possible eigenvector of A if and only if

$$\max_{k \in N} \underline{x}_k \leq m_1 \tag{9}$$

and

$$(\forall r \in S(A))(\forall i \in E^r)[\min_{k \in N_i^r} \overline{x}_k < m_r \Rightarrow \max_{k \in N_i^r} \underline{x}_k \leq \min_{N_i^r} \overline{x}_k].$$

Suppose that (10)

$$\max_{k \in N} \underline{x}_k > m_1 \tag{11}$$

or

$$(\exists r \in S(A))(\exists i \in E^r)[\min_{k \in N_i^r} \overline{x}_k < m_r \wedge \max_{k \in N_i^r} \underline{x}_k > \min_{k \in N_i^r} \overline{x}_k]. \tag{12}$$

Inequality (11) implies the existence of $l \in N$ such that $\underline{x}_l > m_1$. Then for each $x \in \mathbf{X}$ we get $x_l > m_1$. According to Lemma 3.1 there is no eigenvector of A in $\mathbf{X}$. Thus $\mathbf{X}$ is not a possible eigenvector of A.

If condition (12) is fulfilled then there exists $r \in S(A)$, $i \in E^r$ such that $\min_{k \in N_i^r} x_k \leq \min_{k \in N_i^r} \overline{x}_k < m_r$ for each $x \in \mathbf{X}$. Moreover, the inequality $\max_{k \in N_i^r} \underline{x}_k > \min_{k \in N_i^r} \overline{x}_k$ implies the existence of $t, s \in N_i^r$ such that $x_t \geq \underline{x}_t = \max_{k \in N_i^r} \underline{x}_k > \min_{k \in N_i^r} \overline{x}_k = \overline{x}_s \geq x_s$, i.e. $x_t \neq x_s$ for each $x \in \mathbf{X}$. In view of Theorem 3.2 there is no eigenvector $x \in \mathbf{X}$. Thus $\mathbf{X}$ is not a possible eigenvector of A.

For the converse implication suppose that

$$\max_{k \in N} \underline{x}_k \leq m_1 \tag{13}$$

and

$$(\forall r \in S(A))(\forall i \in E^r)[\min_{k \in N_i^r} \overline{x}_k < m_r \Rightarrow \max_{k \in N_i^r} \underline{x}_k \leq \min_{N_i^r} \overline{x}_k]. \tag{14}$$

We shall construct the vector $x^* \in \mathbf{X}$ such that x^* is an eigenvector of A by the following algorithm.

Algorithm $\mathscr{A}_2$: Finding the eigenvector $x^*(A)$

Input: $A(a_0, a_1, \ldots, a_{n-1})$, interval vector $\mathbf{X} \in B(n)$
Output: the sets T^j, the elements $x^*(T^j) \in B$, the numbers $r^*(T^j) \in S(A)$ and the vector $x^* \in B(n)$

begin
 $j := 1$; $\tilde{N} := \emptyset$;
 for $r = s(A) : 1$ **do**
  if $\tilde{N} \neq N$ **then**
   for $i = 1 : e_r$ **do**
    if $\min_{k \in N_i^r} \overline{x}_k < m_r$ and $i \notin \tilde{N}$ **then**
     $T^j := N_i^r$; $r^*(T^j) := r$;
     $x^*(T^j) := \min_{k \in T^j} \overline{x}_k$; (*)
     for $l \in T^j$ **do**
      $x_l^* := x^*(T^j)$; (**)
     enddo
     $\tilde{N} := \tilde{N} \cup T^j$; $j := j + 1$;
    endif
   enddo
  endif
 enddo
end

As the output of Algorithm $\mathscr{A}_2$ we get the sets $T^1, T^2, \ldots, T^p$, $p \leq n$. Similarly as in Algorithm $\mathscr{A}_1$, the sets $T^1, T^2, \ldots, T^p$, $p \leq n$ create the partition $\mathscr{T}$ on N. Moreover, the conditions $i) - iii)$ of Theorem 3.1 are satisfied. The inequality $\max_{k \in N_i^r} \underline{x}_k \leq \min_{N_i^r} \overline{x}_k$ implies that $\min_{k \in N_i^r} \overline{x}_k \in [\underline{x}_l, \overline{x}_l]$ for each $l \in N_i^r$. Thus the constructed vector x^* lies in $\mathbf{X}$.

In view of Theorem 3.1 the vector x^* is an eigenvector of A. Moreover, x^* is the greatest eigenvector in $\mathbf{X}$ because of (*) , (**) and Theorem 3.2. $\square$

4.2. Universal eigenvectors

Theorem 4.2. *Let* $A = A(a_0, a_1, \ldots, a_{n-1})$ *be a circulant matrix. An interval vector* $\mathbf{X}$ *is a universal eigenvector of* A *if and only if* $\underline{x}$ *is an eigenvector of* A*,* $\max_{k \in N} \overline{x}_k \leq m_1$ *and for each* $r \in S(A)$, $i \in N$ *such that* $e_r \neq n$ *and* $\underline{x}_i < m_r$*, the equality* $\underline{x}_i = \overline{x}_i$ *is fulfilled.*

Proof. Suppose that

$\underline{x}$ is not an eigenvector of A or $\max_{k \in N} \overline{x}_k > m_1$

or

$$(\exists i \in N)(\exists r \in S(A))[\underline{x}_i < m_r \wedge e_r \neq n \wedge \underline{x}_i \neq \overline{x}_i]. \quad (15)$$

If $\underline{x}$ is not an eigenvector of A then $\mathbf{X}$ is not a universal eigenvector of A.

In the second case the inequality $\max_{k\in N} \overline{x}_k > m_1$ implies that $\overline{x}$ is not an eigenvector of A and consequently $\mathbf{X}$ is not a universal eigenvector of A.

In the third case suppose that condition (15) is satisfied and $\underline{x}$ is an eigenvector of A. Since $e_r \neq n$, there exists $j \in N,\ j \neq i$ such that $j \equiv i \bmod e_r$. Since $\underline{x}$ is an eigenvector of A, we have $\underline{x}_i = \underline{x}_j$. Define the vector $\tilde{x} \in B(n)$ as follows

$$\tilde{x}_k = \begin{cases} \overline{x}_k & \text{for } k = i, \\ \underline{x}_k & \text{for } k \neq i. \end{cases}$$

Since $\tilde{x}_j < m_r$, $i \equiv j \bmod e_r$ and $\tilde{x}_i \neq \tilde{x}_j$, the vector $\tilde{x}$ is not an eigenvector of A by Lemma 3.2. Thus $\mathbf{X}$ is not a universal eigenvector of A.

For the converse implication suppose that $\underline{x}$ is an eigenvector of A, $\max_{k\in N} \overline{x}_k \leq m_1$ and $\mathbf{X}$ is not a universal eigenvector of A. We will prove that the condition (15) is satisfied.

If $x \in \mathbf{X}$ is not an eigenvector of A and $\max_{k\in N} \overline{x}_k \leq m_1$ then, by Theorem 3.2, there exist $r \in S(A)$ and $j \in E^r$ such that $\min_{k\in N_j^r} x_k < m_r$ and $x_t \neq x_s$ for some $t, s \in N_j^r$. Then $e_r \neq n$ and there exists $i \in N_j^r$ such that $\underline{x}_i = \min_{k\in N_j^r} \underline{x}_k < m_r$. The existence of $t, s \in N_j^r$ such that $x_t \neq x_s$ implies $x_i \neq x_t$ or $x_i \neq x_s$. Without lost of generality we shall suppose that $x_i \neq x_t$. Whereas $\underline{x}$ is an eigenvector of A we get $\underline{x}_i = \underline{x}_t$. In the conjunction with the inequality $x_i \neq x_t$ we get $\underline{x}_i \neq \overline{x}_i$ or $\underline{x}_t \neq \overline{x}_t$. Thus the condition (15) is satisfied. □

Corollary 4.1.

i) *If $A = A(a_0, a_1, \ldots, a_{n-1})$ is such that $P_1 = \{0\}$ and $[\underline{x}_i, \overline{x}_i] \subseteq [m_1, m_2]$ for each $i \in N$ then $\mathbf{X}$ is a universal eigenvector of A.*

ii) *Let for each $i \in N$ the inequality $\underline{x}_i < m_{r^*}$, where $r^* = \min\{r; e_r \neq n\}$, holds true. An interval vector $\mathbf{X}$ is a universal eigenvector of A if and only if $\underline{x} = \overline{x}$ and $\underline{x}$ is an eigenvector of A.*

iii) *If there exists $j \in N$ such that $\underline{x}_j < m_{\tilde{r}}$ where $\tilde{r} = \min\{r; e_r = 1\}$. Then $\mathbf{X}$ is a universal eigenvector of A if and only if $\underline{x}_i = \overline{x}_i = \underline{x}_j$ for each $i \in N$.*

5. EXAMPLES

Example 5.1. *Check out, whether a given interval vector $\mathbf{X}$ is a possible (universal) eigenvector of A, if*

$$A = A(14, 2, 5, 3, 1, 0, 10, 1, 7, 2, 1, 1)$$

and

$$\mathbf{X} = ([4,6],[6,8],[3,10],[5,7],[2,14],[5,12],[4,8],[2,9],$$
$$[3,9],[4,9],[4,10],[9,14])^T.$$

The strictly decreasing sequence of inputs is $M(A) = (m_1, m_2, \ldots, m_8) = (14, 10, 7, 5, 3, 2, 1, 0)$. The sets P_r, $r \in \{1, 2, \ldots, 8\}$ are $P_1 = \{0\}$, $P_2 = \{6\}$, $P_3 = \{8\}$, $P_4 = \{2\}$, $P_5 = \{3\}$, $P_6 = \{1, 9\}$, $P_7 = \{4, 7, 10, 11\}$, $P_8 = \{5\}$. We compute the numbers e_r, $r \in \{1, 2, \ldots, 8\}$ using (3). We get $e_1 = 12$, $e_2 = 6$, $e_3 = 2$, $e_4 = 2$, $e_5 = e_6 = e_7 = e_8 = 1$.

We have to check conditions (9) and (10). Denote by (I) the implication

$$\min_{k\in N_i^r} \overline{x}_k < m_r \Rightarrow \max_{k\in N_i^r} \underline{x}_k \leq \min_{N_i^r} \overline{x}_k \quad \text{(I)}$$

from (10).

Since $\max_{k\in N} \underline{x}_k = 9 \leq m_1$ inequality (9) holds true. To verify (10) we start with $r = s(A) = 8$. We compute $e_8 = 1$, $E_8 = \{1\}$ and $N_1^8 = N$. Since $\min_{k\in N} \overline{x}_k = 6 \geq 0$ the implication (I) is true for $r = 8$.

In fact, it is sufficient to start with the least r such that $e_r = 1$, i.e., $r = 5$. Since $\min_{k\in N} \overline{x}_k = 6 \geq m_5$ the implication in (10) is true for $r \in \{5, 6, 7, 8\}$.

Similarly, since $e_3 = e_4 = 2$ we continue with $r = 3$. We have $E_3 = \{1, 2\}$ and $N_1^3 = \{1, 3, 5, 7, 9, 11\}$, $N_2^3 = \{2, 4, 6, 8, 10, 12\}$.

For $i = 1$ we have $\min_{k\in N_1^3} \overline{x}_k = 6 < m_3$, so we compute $\max_{k\in N_1^3} \underline{x}_k = 4 \leq \min_{k\in N_1^3} \overline{x}_k$. Thus the implication (I) holds true. Moreover, we can construct the vector x^* using Algorithm $\mathscr{A}_2$. We put $x_1^* = x_3^* = x_5^* = x_7^* = x_9^* = x_{11}^* = 6$. We have $\tilde{N} = N_1^3$.

For $i = 2$ we compute $\min_{k\in N_2^3} \overline{x}_k = 8 \geq m_3$, hence the implication is true.

For $r = 2$ we have $N_1^2 = \{1, 7\}$, $N_2^2 = \{2, 8\}$, $N_3^2 = \{3, 9\}$, $N_4^2 = \{4, 10\}$, $N_5^2 = \{5, 11\}$, $N_6^2 = \{6, 12\}$. For $i \in \{1, 3, 5\}$ we have $i \in \tilde{N}$, so we consider only $i \in \{2, 4, 6\}$. We compute $\min_{k\in N_2^2} \overline{x}_k = 8 < m_2$ and $\max_{k\in N_2^2} \underline{x}_k = 6 \leq \min_{k\in N_2^2} \overline{x}_k$. We put $x_2^* = x_8^* = 8$.

For the set N_4^2 we get $\min_{k\in N_4^2} \overline{x}_k = 7 < m_2$ and $\max_{k\in N_4^2} \underline{x}_k = 5 \leq \min_{k\in N_4^2} \overline{x}_k$. We put $x_4^* = x_{10}^* = 7$.

Since $\min_{k\in N_6^2} \overline{x}_k = 12 \geq m_2$, the implication (I) is true.

For $r = 1$ we have $N_i^1 = \{i\}$ for each $i \in N$. In this case the implication always holds true. It remains to put $x_6^* = 12$, $x_{12}^* = 14$.

The interval vector $\mathbf{X}$ is not a universal eigenvector of A, because the vector $\underline{x}$ is not an eigenvector of A.

Answer:

- The interval vector $\mathbf{X}$ is a possible eigenvector of A and the vector $x^* = (6, 8, 6, 7, 6, 12, 6, 8, 6, 7, 6, 14)^T$ is the greatest eigenvector of A lying in $\mathbf{X}$.
- The interval vector $\mathbf{X}$ is not a universal eigenvector of A.

Example 5.2. *In this example we show three universal eigenvectors of the matrix $A(14, 2, 5, 3, 1, 0, 10, 1, 7, 2, 1, 1)$ from the previous example using Corollary 4.1.*

i) The given matrix satisfies the condition $P^1 = \{0\}$. We take the vector $\mathbf{X}$ such that $[\underline{x}_i, \overline{x}_i] \subseteq [m_1, m_2] = [10, 14]$ for each $i \in N$. This condition is satisfied, e.g., by interval vector

$$\mathbf{X} = ([12,14],[11,13],[10,14],[11,12],[13,13],[10,12],$$

$$[12,13],[11,12],[10,13],[11,11],[12,14],[11,12]).$$

In view of Corollary 4.1i) the vector $\mathbf{X}$ is a universal eigenvector of A.

ii) To demonstrate the use of Corollary 4.1ii), let us set $\underline{x} = (5,7,5,7,5,8,5,7,5,7,5,8)^T$. The vector $\underline{x}$ is an eigenvector of A according to Theorem 3.2. Since $r^* = 2$ and $\underline{x}_i < m_2 = 10$ for each $i \in N$, the only universal eigenvector $\mathbf{X}$ is the constant vector, i.e., $\overline{x} = \underline{x}$.

iii) We have $\tilde{r} = 5$, $m_{\tilde{r}} = 3$. Let $\underline{x}_1 = 2$. According to Corollary 4.1iii), the only universal eigenvector of A is such that $\underline{x}_i = \overline{x}_i = 2$ for each $i \in N$.

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