ON THE CROSSING NUMBER OF THE JOIN OF FIVE VERTEX GRAPH G WITH THE DISCRETE GRAPH D_n

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ABSTRACT

In this paper, we show the values of crossing numbers for join products of graph G on five vertices with the discrete graph D_n and the path P_n on n vertices. The proof is done with the help of software. The software generates all cyclic permutations for a given number n. For cyclic permutations, $P_1 - P_m$ will create a graph in which to calculate the distances between all vertices of the graph. These distances are used in proof of crossing numbers of presented graphs.

Keywords: *crossing number, cyclic permutations, drawing, graph, join*

1. INTRODUCTION

Let G be a simple graph with the vertex set V and the edge set E. A *drawing* of the graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. In such a drawing, the intersection of the interiors of the arcs is called a *crossing*. We assume that in a drawing no edge passes through any vertex other than its end-points, no two edges touch each other (i.e., if two edges have a common interior point, then they cross properly at this point), and no three edges cross at the same point. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross each other.

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is defined as the minimum possible number of edge crossings in a drawing of G in the plane.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be simple graphs. The *join product* of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$, and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$.

Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $cr_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

In the paper, some proofs are based on the Kleitman's result on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

2. SUBJECT

2.1. Software description

We will describe in this subchapter the software which we use when proving the Theorem 4.1 in this article and also in a similar proofs of theorems such like this. The input for the algorithm is the number n, which represents an n-element set $\{1, 2, 3, \ldots, n\}$. The algorithm selects all cyclic permutations from the set of all permutations of the n-element set $\{1, 2, 3, \ldots, n\}$. The software marks these permutations with symbols $P_1, \ldots, P_m$, where $m = (n-1)!$. Said software gives outputs of distance between each pair of vertices of given graph.

A graph is created with a set of vertices $V = \{P_1, P_2, \ldots, P_m\}$ and set of edges E, where the two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two elements of the n-tuple (i. e. an ordered set with n elements). This graph is represented by a square symmetrical adjacency matrix. The distance between each pair of vertices are calculated using the properties of the cyclic-order graph CO_5 defined in [5].

The software uses the following graph theory:
Let us denote $B^{(1)}$ the matrix is gotten from adjacency matrix B by adding ones to the main diagonal. Let us consider the matrix $B^{(2)} = \{b_{ij}^{(2)}\}_{i,j=1}^{m}$ such that $B^{(2)} = B^{(1)} \cdot B^{(1)}$. From the matrix multiplication it is obvious that $b_{ij}^{(2)} = \sum_{k=1}^{m} b_{ik}^{(1)} \cdot b_{kj}^{(1)}$, but in this matrix we will use the Boolean addition and multiplication ($1 \cdot 1 = 1$, $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$, $1+0 = 0+1 = 1+1 = 1$, and $0+0 = 0$). Generally we can consider matrix $B^{(m)} = B^{(m-1)} \cdot B^{(1)}$.

Theorem 2.1. *Let the adjacency matrix B of the connected graph $G = (V, H)$, $|V| = n$ is given. Then for arbitrary $k = 1, 2, \ldots, m$, the element $b_{ij}^{(k)}$ of the matrix $B^{(k)}$ is equal to one if $d(v_i, v_j) \leq k$.*

Corollary 2.1. *The graph $G = (V, H)$, $|V| = n$, is con-*

nected only when the elements of the matrix $B^{(n-1)}$ are only ones.

Corollary 2.2. *For each two different vertices of the graph $G = (V, H)$, $d(v_i, v_j) = \min_{k\in\{1,2,\dots,m\}}\{k; b_{ij}^{(k)} = 1\}$.*

2.2. Important facts

We will show the correct proof of the theorem from the article [1]. We deal with the graph G with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ which is shown in Fig. 1. There is also the graph G with renamed vertices $V = \{1,2,3,4,5\}$, which is done counter-clockwise with the beginning in the upper right corner.

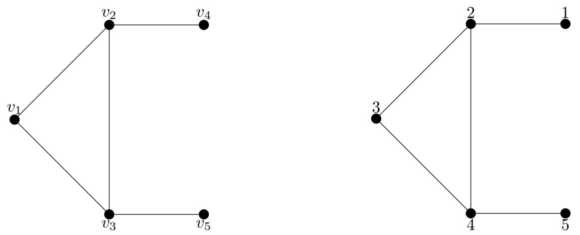

Fig. 1 Five vertex graph G and its numbering of vertices

Using the given software for $n = 5$, we get the names of the cyclic permutations (see Table 1) and the distance between the cyclic permutations (see Table 2 - 5). The maximum distance between two vertices in the graph is equal to four. With this information we can use arguments which are described in [3,4].

Table 1 Names of Cyclic Permutations of 5-elements set

Name	Cyclic perm.	Name	Cyclic perm.
$P_1 \longrightarrow$	(1 2 3 4 5)	$P_{13} \longrightarrow$	(1 2 5 4 3)
$P_2 \longrightarrow$	(1 3 2 4 5)	$P_{14} \longrightarrow$	(1 5 2 4 3)
$P_3 \longrightarrow$	(1 2 4 3 5)	$P_{15} \longrightarrow$	(1 2 4 5 3)
$P_4 \longrightarrow$	(1 4 2 3 5)	$P_{16} \longrightarrow$	(1 4 2 5 3)
$P_5 \longrightarrow$	(1 4 3 2 5)	$P_{17} \longrightarrow$	(1 4 5 2 3)
$P_6 \longrightarrow$	(1 3 4 2 5)	$P_{18} \longrightarrow$	(1 5 4 2 3)
$P_7 \longrightarrow$	(1 2 3 5 4)	$P_{19} \longrightarrow$	(1 5 3 4 2)
$P_8 \longrightarrow$	(1 3 2 5 4)	$P_{20} \longrightarrow$	(1 3 5 4 2)
$P_9 \longrightarrow$	(1 2 5 3 4)	$P_{21} \longrightarrow$	(1 5 4 3 2)
$P_{10} \longrightarrow$	(1 5 2 3 4)	$P_{22} \longrightarrow$	(1 4 5 3 2)
$P_{11} \longrightarrow$	(1 5 3 2 4)	$P_{23} \longrightarrow$	(1 4 3 5 2)
$P_{12} \longrightarrow$	(1 3 5 2 4)	$P_{24} \longrightarrow$	(1 3 4 5 2)

3. METHODS

We consider the join of G with the discrete graph on n vertices D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \le i \le n$, denote the subgraph induced by the five edges incident with the vertex t_i. Then

$$G + D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i\right).$$

The graph $G + D_1$ is planar, thus $\mathrm{cr}(G + D_1) = 0$. One can easy to verify that $\mathrm{cr}(G + D_2) \le 1$. The graph $G + D_2$ contains a subdivision of $K_{3,3}$ as a subgraph, and therefore $\mathrm{cr}(G + D_2) \ge 1$. So, $\mathrm{cr}(G + D_2) = 1$.

Let D be a good drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of vertex a t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [4]. We use the notation (12543) if the counter-clockwise order the edges incident with the vertex t_i is t_i1, t_i2, t_i5, t_i4, and t_i3 (see A_1 in Fig. 2), where $V = \{1,2,3,4,5\}$ are noted vertices of the graph G. We emphasize that a rotation is a cyclic permutation. For $i, j \in \{1, 2, \dots, n\}$, $i \ne j$, every subgraph $T^i \cup T^j$ of the graph $G + D_n$ is isomorphic with the graph $K_{5,2}$. In the paper, we will deal with the minimum necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced by the drawing D of the graph $G + D_n$ depending on the rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$.

D. R. Woodall [5] proved that, in any good drawing D of the graph $K_{5,2}$, $\mathrm{cr}_D(T^i, T^j) \ge 4$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$, then $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \le \mathrm{cr}_D(T^i, T^j)$.

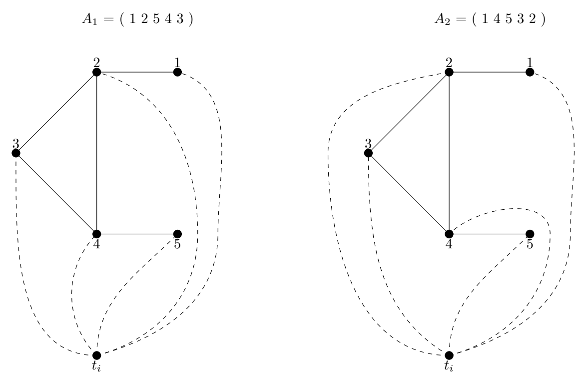

Fig. 2 Graph G and its configurations - type A

We will separate the subgraphs T^i, $i = 1, \dots, n$, of $G + D_n$ into three subsets depending on how many the considered T^i crosses the edges of G in D. For $i = 1, 2, \dots, n$, we denote by $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, any F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

Table 2 Distance 1 Between Cyclic Permutations

From	To
P_1	$P_2, P_3, P_7, P_{10}, P_{24}$
P_2	$P_1, P_6, P_8, P_{11}, P_{15}$
P_3	$P_1, P_4, P_{14}, P_{15}, P_{23}$
P_4	$P_3, P_5, P_7, P_{16}, P_{18}$
P_5	$P_4, P_6, P_8, P_{21}, P_{23}$
P_6	$P_2, P_5, P_{16}, P_{19}, P_{24}$
P_7	$P_1, P_4, P_8, P_9, P_{20}$
P_8	$P_2, P_5, P_7, P_{12}, P_{13}$
P_9	$P_7, P_{10}, P_{13}, P_{16}, P_{19}$
P_{10}	$P_1, P_9, P_{11}, P_{14}, P_{17}$
P_{11}	$P_2, P_{10}, P_{12}, P_{19}, P_{22}$
P_{12}	$P_8, P_{11}, P_{14}, P_{20}, P_{23}$
P_{13}	$P_8, P_9, P_{14}, P_{15}, P_{21}$
P_{14}	$P_3, P_{10}, P_{12}, P_{13}, P_{18}$
P_{15}	$P_2, P_3, P_{13}, P_{16}, P_{22}$
P_{16}	$P_4, P_6, P_9, P_{15}, P_{17}$
P_{17}	$P_{10}, P_{16}, P_{18}, P_{22}, P_{24}$
P_{18}	$P_4, P_{14}, P_{17}, P_{20}, P_{21}$
P_{19}	$P_6, P_9, P_{11}, P_{20}, P_{21}$
P_{20}	$P_7, P_{12}, P_{18}, P_{19}, P_{24}$
P_{21}	$P_5, P_{13}, P_{18}, P_{19}, P_{22}$
P_{22}	$P_{11}, P_{15}, P_{17}, P_{21}, P_{23}$
P_{23}	$P_3, P_5, P_{12}, P_{22}, P_{24}$
P_{24}	$P_1, P_6, P_{17}, P_{20}, P_{23}$

Table 3 Distance 2 Between Cyclic Permutations

From	To
P_1	$P_4, P_6, P_8, P_9, P_{11}, P_{14}, P_{15}, P_{17}, P_{20}, P_{23}$
P_2	$P_3, P_5, P_7, P_{10}, P_{12}, P_{13}, P_{16}, P_{19}, P_{22}, P_{24}$
P_3	$P_2, P_5, P_7, P_{10}, P_{12}, P_{13}, P_{16}, P_{18}, P_{22}, P_{24}$
P_4	$P_1, P_6, P_8, P_9, P_{14}, P_{15}, P_{17}, P_{20}, P_{21}, P_{23}$
P_5	$P_2, P_3, P_7, P_{12}, P_{13}, P_{16}, P_{18}, P_{19}, P_{22}, P_{24}$
P_6	$P_1, P_4, P_8, P_9, P_{11}, P_{15}, P_{17}, P_{20}, P_{21}, P_{23}$
P_7	$P_2, P_3, P_5, P_{10}, P_{12}, P_{13}, P_{16}, P_{18}, P_{19}, P_{24}$
P_8	$P_1, P_4, P_6, P_9, P_{11}, P_{14}, P_{15}, P_{20}, P_{21}, P_{23}$
P_9	$P_1, P_4, P_6, P_8, P_{11}, P_{14}, P_{15}, P_{17}, P_{20}, P_{21}$
P_{10}	$P_2, P_3, P_7, P_{12}, P_{13}, P_{16}, P_{18}, P_{19}, P_{22}, P_{24}$
P_{11}	$P_1, P_6, P_8, P_9, P_{14}, P_{15}, P_{17}, P_{20}, P_{21}, P_{23}$
P_{12}	$P_2, P_3, P_5, P_7, P_{10}, P_{13}, P_{18}, P_{19}, P_{22}, P_{24}$
P_{13}	$P_2, P_3, P_5, P_7, P_{10}, P_{12}, P_{16}, P_{18}, P_{19}, P_{22}$
P_{14}	$P_1, P_4, P_8, P_9, P_{11}, P_{15}, P_{17}, P_{20}, P_{21}, P_{23}$
P_{15}	$P_1, P_4, P_6, P_8, P_9, P_{11}, P_{14}, P_{17}, P_{21}, P_{23}$
P_{16}	$P_2, P_3, P_5, P_7, P_{10}, P_{13}, P_{18}, P_{19}, P_{22}, P_{24}$
P_{17}	$P_1, P_4, P_6, P_9, P_{11}, P_{14}, P_{15}, P_{20}, P_{21}, P_{23}$
P_{18}	$P_3, P_5, P_7, P_{10}, P_{12}, P_{13}, P_{16}, P_{19}, P_{22}, P_{24}$
P_{19}	$P_2, P_5, P_7, P_{10}, P_{12}, P_{13}, P_{16}, P_{18}, P_{22}, P_{24}$
P_{20}	$P_1, P_4, P_6, P_8, P_9, P_{11}, P_{14}, P_{17}, P_{21}, P_{23}$
P_{21}	$P_4, P_6, P_8, P_9, P_{11}, P_{14}, P_{15}, P_{17}, P_{20}, P_{23}$
P_{22}	$P_2, P_3, P_5, P_{10}, P_{12}, P_{13}, P_{16}, P_{18}, P_{19}, P_{24}$
P_{23}	$P_1, P_4, P_6, P_8, P_{11}, P_{14}, P_{15}, P_{17}, P_{20}, P_{21}$
P_{24}	$P_2, P_3, P_5, P_7, P_{10}, P_{12}, P_{16}, P_{18}, P_{19}, P_{22}$

Table 4 Distance 3 Between Cyclic Permutations

From	To
P_1	$P_5, P_{12}, P_{13}, P_{16}, P_{18}, P_{19}, P_{22}$
P_2	$P_4, P_9, P_{14}, P_{17}, P_{20}, P_{21}, P_{23}$
P_3	$P_6, P_8, P_9, P_{11}, P_{17}, P_{20}, P_{21}$
P_4	$P_2, P_{10}, P_{12}, P_{13}, P_{19}, P_{22}, P_{24}$
P_5	$P_1, P_9, P_{11}, P_{14}, P_{15}, P_{17}, P_{20}$
P_6	$P_3, P_7, P_{10}, P_{12}, P_{13}, P_{18}, P_{22}$
P_7	$P_6, P_{11}, P_{14}, P_{15}, P_{17}, P_{21}, P_{23}$
P_8	$P_3, P_{10}, P_{16}, P_{18}, P_{19}, P_{22}, P_{24}$
P_9	$P_2, P_3, P_5, P_{12}, P_{18}, P_{22}, P_{24}$
P_{10}	$P_4, P_6, P_8, P_{15}, P_{20}, P_{21}, P_{23}$
P_{11}	$P_3, P_5, P_7, P_{13}, P_{16}, P_{18}, P_{24}$
P_{12}	$P_1, P_4, P_6, P_9, P_{15}, P_{17}, P_{21}$
P_{13}	$P_1, P_4, P_6, P_{11}, P_{17}, P_{20}, P_{23}$
P_{14}	$P_2, P_5, P_7, P_{16}, P_{19}, P_{22}, P_{24}$
P_{15}	$P_5, P_7, P_{10}, P_{12}, P_{18}, P_{19}, P_{24}$
P_{16}	$P_1, P_8, P_{11}, P_{14}, P_{20}, P_{21}, P_{23}$
P_{17}	$P_2, P_3, P_5, P_7, P_{12}, P_{13}, P_{19}$
P_{18}	$P_1, P_6, P_8, P_9, P_{11}, P_{15}, P_{23}$
P_{19}	$P_1, P_4, P_8, P_{14}, P_{15}, P_{17}, P_{23}$
P_{20}	$P_2, P_3, P_5, P_{10}, P_{13}, P_{16}, P_{22}$
P_{21}	$P_2, P_3, P_7, P_{10}, P_{12}, P_{16}, P_{24}$
P_{22}	$P_1, P_4, P_6, P_8, P_9, P_{14}, P_{20}$
P_{23}	$P_2, P_7, P_{10}, P_{13}, P_{16}, P_{18}, P_{19}$
P_{24}	$P_4, P_8, P_9, P_{11}, P_{14}, P_{15}, P_{21}$

Table 5 Distance 4 Between Cyclic Permutations

From	To
P_1	P_{21}
P_2	P_{18}
P_3	P_{19}
P_4	P_{11}
P_5	P_{10}
P_6	P_{14}
P_7	P_{22}
P_8	P_{17}
P_9	P_{23}
P_{10}	P_5
P_{11}	P_4
P_{12}	P_{16}
P_{13}	P_{24}
P_{14}	P_6
P_{15}	P_{20}
P_{16}	P_{12}
P_{17}	P_8
P_{18}	P_2
P_{19}	P_3
P_{20}	P_{15}
P_{21}	P_1
P_{22}	P_7
P_{23}	P_9
P_{24}	P_{13}

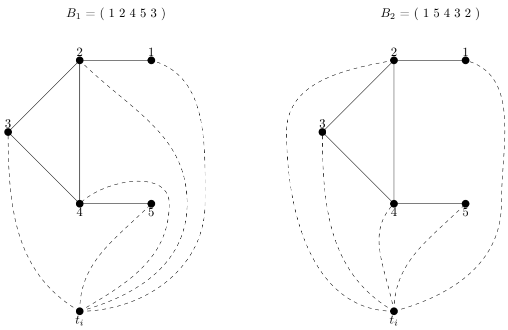

Fig. 3 Graph G and its configurations - type B

There is only one drawing of G without crossings shown in Fig. 1. Assume a good drawing D of the graph $G+D_n$ in which the edges of G does not cross each other. We will count the number of necessary crossings between two subgraphs T^i and T^j with $\mathrm{cr}_D(G, T^i \cup T^j) = 0$. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Fig. 1. It is easy to see that, in D, there are only four different possible configurations of F^i summarized in Table 6, see Fig. 2 and 3. We denote by $\mathscr{M}_D$ the set of all configurations that exist in the drawing D belonging to the set $\mathscr{M}$, where $\mathscr{M} = \{A_1, A_2, B_1, B_2\}$.

Table 6 Configurations of graph $G \cup T^i$ with vertices denoted of G as in Fig. 1

$A_1 : (12543)$	$A_2 : (14532)$
$B_1 : (12453)$	$B_2 : (15432)$

Let X, Y be configurations from $\mathscr{M}_D$. We shortly denote by $\mathrm{cr}_D(X,Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X,Y) = \min\{\mathrm{cr}_D(X,Y)\}$ over all good drawings D of the graph $G+D_n$ with $X, Y \in \mathscr{M}_D$. The configuration A_1 is represented by the cyclic permutation $P_{13} = (12543)$ and the configuration A_2 is represented by the cyclic permutation $P_{22} = (14532)$. As $P_7 = (12354)$ is the inverse cyclic permutation to the permutation P_{22}, then $\mathrm{cr}(A_1, A_2) \geq 2$ by Table 3. The similar idea is used for the another cases. Thus, all lower-bounds of numbers of crossings of configurations from $\mathscr{M}$ are summarized in Table 7.

Table 7 Lower-bounds of numbers of crossings of two configurations from $\mathscr{M}$

	A_1	A_2	B_1	B_2
A_1	4	2	3	3
A_2	2	4	3	3
B_1	3	3	4	2
B_2	3	3	2	4

4. RESULTS

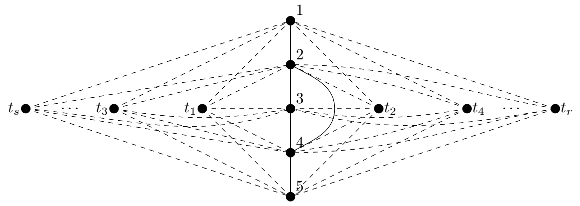

Fig. 4 Good drawing of $G+D_n$

Theorem 4.1. *Let G be the graph in Fig. 1 and D_n is discrete graph with n vertices, then*

$\mathrm{cr}(G+D_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$, *for* $n \geq 1$.

Proof: The theorem is true for $n=1$ and $n=2$. In Fig. 4 there is a drawing of $G+D_n$ with $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ crossings. Thus, $\mathrm{cr}(G+D_n) \leq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$. We prove the reverse inequality by induction on n. For $n \geq 3$, let D be a good drawing of $G+D_n$ with less than $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ crossings. Suppose now that, for $n \geq 3$

$$\mathrm{cr}(G+D_{n-2}) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + \left\lfloor\frac{n-2}{2}\right\rfloor$$

and consider such a drawing D of $G+D_n$ that

$$\mathrm{cr}_D(G+D_n) < 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor. \qquad (1)$$

The drawing D has the following property:

$$\mathrm{cr}_D(T^i, T^j) \neq 0 \qquad \text{for all } i,j = 1,2,\dots,n,\ i \neq j. \qquad (2)$$

To prove it assume that there are two different subgraphs T^i and T^j such that $\mathrm{cr}_D(T^i, T^j) = 0$ and let for every integer s, $s < n$, any good drawing of graph $G+D_s$ has at least $4\left\lfloor\frac{s}{2}\right\rfloor\left\lfloor\frac{s-1}{2}\right\rfloor + \left\lfloor\frac{s}{2}\right\rfloor$ crossings. Without loss of generality let $\mathrm{cr}_D(T^{n-1}, T^n) = 0$, one can easy to verify that $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 1$. By $\mathrm{cr}(K_{5,3}) = 4$ we give $\mathrm{cr}_D(T^k, T^{n-1} \cup T^n) \geq 4$ for $k = 1,2,\dots,n-2$. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D\left(G \cup \bigcup_{i=1}^{n-2} T^i\right) + \mathrm{cr}_D(T^{n-1} \cup T^n) +$$

$$+\mathrm{cr}_D(G, T^{n-1} \cup T^n) + \mathrm{cr}_D\left(\bigcup_{i=1}^{n-2} T^i, T^{n-1} \cup T^n\right) \geq$$

$$\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + \left\lfloor\frac{n-2}{2}\right\rfloor + 1 + 4(n-2) =$$

$$= 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

This contradicts (1), and therefore $\mathrm{cr}_D(T^i, T^j) \neq 0$ for all $i,j = 1,2,\dots,n$, $i \neq j$. Our assumption on D together with $\mathrm{cr}(K_{5,n}) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{5,n}) < \left\lfloor\frac{n}{2}\right\rfloor.$$

Thus, we have $r = |R_D| > \left\lfloor\frac{n}{2}\right\rfloor$, $s = |S_D| < \left\lfloor\frac{n}{2}\right\rfloor$.

Case 1: $\mathrm{cr}_D(G) = 0$ as in Fig. 1.

a) $\{A_1, A_2\} \subseteq \mathscr{M}_D$ or $\{B_1, B_2\} \subseteq \mathscr{M}_D$.
Without lost of generality if we fix any two T^n, $T^{n-1} \in R_D$ such that F^n, F^{n-1} have configurations A_1, A_2, respectively, then $\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^i) \geq 6$ holds by Table 7 for any $T^i \in R_D$. Using (2) we have

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(G \cup T^n \cup T^{n-1}) +$$
$$+\mathrm{cr}_D(K_{5,n-2}, G \cup T^n \cup T^{n-1}) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor +$$
$$+6(r-2) + 3s + 4(n-r-s) + 2 =$$
$$= 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2r - s + 4n - 10 \geq$$
$$\geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) +$$
$$+1 - \left\lfloor \frac{n}{2} \right\rfloor + 4n - 10 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

b) $\{A_1, A_2\} \not\subseteq \mathscr{M}_D$ and $\{B_1, B_2\} \not\subseteq \mathscr{M}_D$.
Without lost of generality if we fix any $T^n \in R_D$ such that F^n has configuration from $\mathscr{M}_D$, then $\mathrm{cr}_D(G \cup T^n, T^i) \geq 3$ holds for any $T^i \in R_D$. Using the property (2) we have

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(G \cup T^n) +$$
$$+\mathrm{cr}_D(K_{5,n-1}, G \cup T^n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor +$$
$$+3(r-1) + 2s + 3(n-r-s) =$$
$$= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n - s - 3 \geq$$
$$\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 1 - \left\lfloor \frac{n}{2} \right\rfloor - 3 \geq$$
$$\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor.$$

In the following three cases we will use the same idea as in the Case 1 b). Since $r = |R_D| > \left\lfloor \frac{n}{2} \right\rfloor$, the vertices of degree one noted by 1, 5 cannot be separated by 3-cycle of the graph G.

Case 2: $\mathrm{cr}_D(G) = 1$ as in Fig. 5.

By a discussion we can easy verify that there is only one type of configuration for F^i represented by $P_5 = (14325)$.

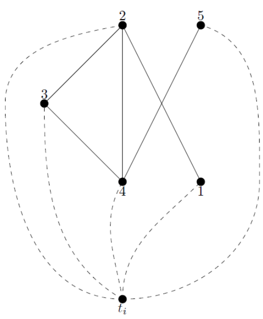

Fig. 5 Good drawing of $G \cup T^i$

Case 3: $\mathrm{cr}_D(G) = 1$ as in Fig. 6.

By a discussion we can verify that there are only two type of configurations for F^i represented by the cyclic permutations $P_{23} = (14352)$ and $P_3 = (12435)$. $P_{19} = (15342)$ is the inverse cyclic permutation to the permutation P_3. Thus, by Table 4 we give lower-bound of number of crossings of these configurations equal to three.

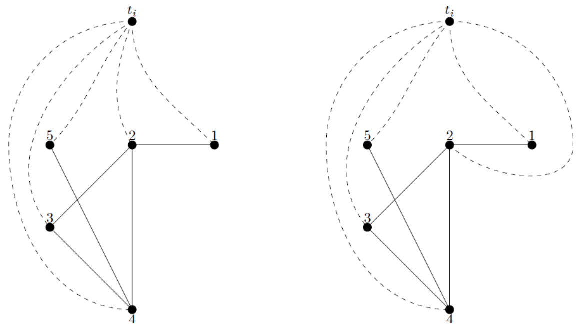

Fig. 6 Good drawing of $G \cup T^i$

Case 4: $\mathrm{cr}_D(G) = 3$ as in Fig. 7.

By a discussion we can verify that there is only one type of configuration for F^i represented by the cyclic permutation $P_{12} = (13524)$.

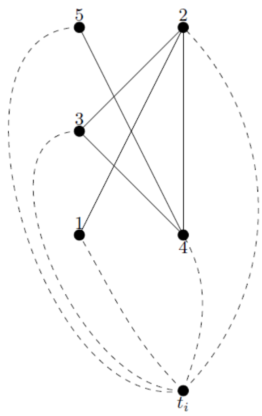

Fig. 7 Good drawing of $G \cup T^i$

Theorem 4.2. *Let G be the graph in Fig. 1 and P_n is a path on n vertices, then*

$$\mathrm{cr}(G+P_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \quad \textit{for} \quad n \geq 1.$$

We are able to add the edges without crossings in Fig. 4. So the drawing of the graph $G+P_n$ with $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ crossings is obtained.

5. CONCLUSIONS

In this article, we show the proof technique for a crossing number in a given graphs that used the data generated by the software. More significant usage of this software occurs for larger values of n than five. We get 120 cyclical permutations for $n = 6$, 720 cyclical permutations for $n = 7$ which is significantly more than 24 cyclical permutations for $n = 5$. For such values, software is an indispensable tool since, we get considerably more complicated graph of distances between cyclic permutations.

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